Non-gaussian geometrical measures in redshift space

Dmitry Pogosyan

Physics Department
University of Alberta

April 10, 2019

with: S. Codis, C. Pichon, F. Bernardeau, T. Matsubara

Density in redshift space

$$\rho_s(\mathbf{X}, v) = \int dz\, \rho(\mathbf{x})\, \frac{1}{\sqrt{2\pi\beta_T(\mathbf{x})}} \exp\left[-\frac{(v - v_c(\mathbf{x}) - u(\mathbf{x}))^2}{2\beta_T(\mathbf{x})} \right]$$

Cold flow: $\beta_T \rightarrow 0$ (effectivly never achieved due to finite redshift rezolution)

$$\rho_s(\mathbf{X}, v) = \int dz\, \rho(\mathbf{x})\, \delta_D\left(v - v_c(\mathbf{x}) - u(\mathbf{x})\right)$$

ISM/turbulence studies

No detailed predictions from underlying theory: mode content or relation between velocities and density are not known apriori.

Lesson: slice and dice PPV cube, synthetically varying some control parameters to disentangle effects.

Example: measuremets in velocity channels of variable width.

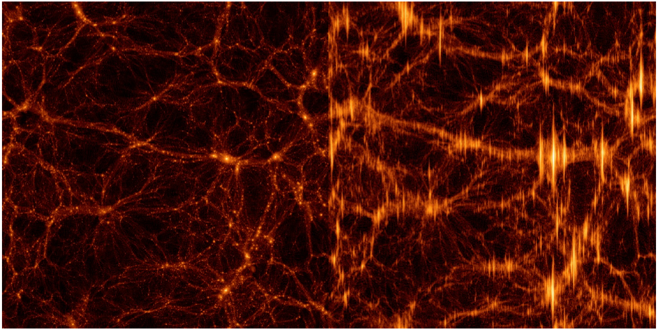
Cosmology

Good predictive understanding of theory that lead to PPZ.

Natural approach: try to fit full predictions of power and higher order spectra.

Still valuable - measure some integrals of spectra that take us straight to cosmological info. Having control parameters to vary in your analysis helps.

Geometrical measures for random fields



- Properties of $\nu = const$ isocontours - Minkowski functionals (genus/Euler characteristics $\chi(\nu)$, length of isocontours, pencil beam isocontour crossing statistics, $N_D(\nu)$)
- Statistics of extrema - peaks, minima, saddles
- Skeleton of the structure, its statistics (length, curvature)
- ...

Geometrical measures for random fields

Starting point

Let us think about such properties of a random field ρ as Euler characteristic (genus), density of maxima, length of skeleton. Their computation reguire knowledge of the joint distribution

$$\mathscr{P}(\rho, \rho_i, \rho_{ij}, \ldots)$$

of the field ρ and its first ρ_i, second ρ_{ij} (Hessian matrix) and perhaps higher derivatives, for instance

$$n_{max}(\nu) = \int_{0 \geq \lambda_1 \geq \lambda_2 \geq \ldots} \mathscr{P}(\rho = \nu, \rho_i, \rho_{ij}) \delta(\rho_i) |\rho_{ij}| d\rho_{ij}$$

Usual approach is to deal with it in the Hessian eigenvalue space, since that's where the boundary conditions are the simplest.

Non-Gaussian expansion for geometrical statistics

- To treat non-Gaussianities the idea is to expand $\mathscr{P}(\rho,\rho_i,\rho_{ij})$ into orthogonal polinomials around the Gaussian approximation like
$$P(x)=G(x)\left(1+\sum_n\langle x^n\rangle_c H_n(x)\right)$$
- The trick to avoid difficulties is an appropriate choice of variables:
  - that are invariant wrt symmetries of the problem (isotropy)
  - that are polynomial in the field quanities (λ's are no good)
  - that simplify the Gaussian limit, being as uncorrelated as possible
- Useful set is: $I_1,\cdots,I_N,\ q^2,\ \zeta\equiv\frac{\rho+\gamma I_1}{1-\gamma^2}$
where I_n are N polynomial rotation invariants of the Hessian matrix ρ_{ij},
$$I_1=\mathrm{Tr}\,\rho_{ij}\quad,\ \ldots\ ,\quad I_N=\det|\rho_{ij}|$$
(and $I_2\ldots I_{N-1}$ are built from the minors of orders 2 to N-1)
- Actually, better to use more 'irreducible' combinations J_i, in ND-space

$$J_1=I_1\ ,\quad J_{s\geq 2}=I_1^s-\sum_{p=2}^{s}\frac{(-N)^p C_s^p}{(s-1)C_N^p}I_1^{s-p}I_p$$

Orhogonal polymonial expansion for 2D $\mathscr{P}(\rho, q^2, J_1, J_2)$

Gaussian limit JPDF

$$G_{\rm 2D}(\zeta, q^2, J_1, J_2)\, \mathrm{d}\zeta \mathrm{d}q^2 \mathrm{d}J_1 \mathrm{d}J_2 = \frac{1}{2\pi} e^{-\frac{1}{2}(\zeta^2 + 2q^2 + J_1^2 + 2J_2)} \mathrm{d}\zeta \mathrm{d}q^2 \mathrm{d}J_1 \mathrm{d}J_2$$

serves as the weight for defining the expansion polynomials in

- ζ, J_1 – ($[-\infty, \infty]$, gaussian weight) – Hermite
- q^2, J_2 – ($[0, \infty]$, exponential weight) – Laguerre.

$$\mathscr{P}_{\rm 2D}(\zeta, q^2, J_1, J_2) = G_{\rm 2D} \times \Bigg[1 + \sum_{n=3}^{\infty} \sum_{i,j,k,l=0}^{i+2j+k+2l=n} \frac{(-1)^{j+l}}{i!\, j!\, k!\, l!} \left\langle \zeta^i q^{2j} J_1{}^k J_2{}^l \right\rangle_{\rm GC} H_i(\zeta) L_j\left(q^2\right) H_k(J_1) L_l(J_2) \Bigg]$$

This is an expansion to all orders in powers of the field n.

Expansion coefficients can be predicted by pertrubation theories

Euler characteristic (genus) as a function of threshold

General expression in ND

$$\frac{\chi(\nu)}{2} = (-1)^N \int_\nu^\infty dx \int dq^2 q^{N-1} \delta_D^N(q^2) \int \prod_{s=1}^{N} dJ_s \mathscr{P}_{\mathrm{ND}}(\ldots)\, I_N$$

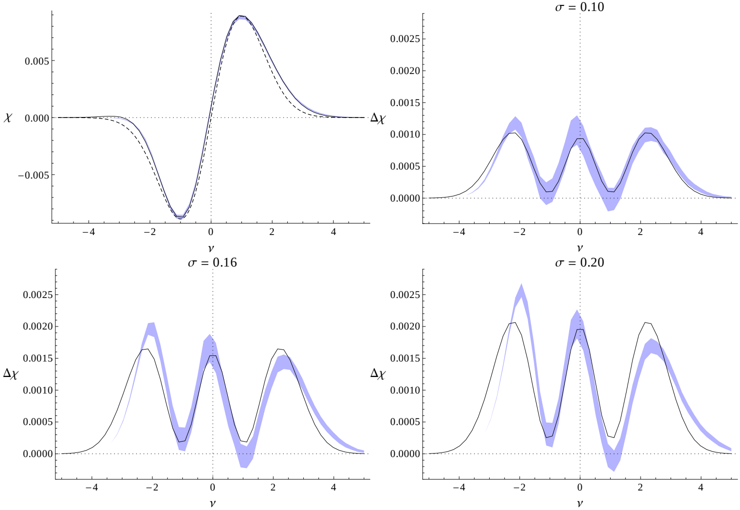
can be integrated to give "moment" expansion to all orders

$$\chi(\nu) = \frac{1}{\sqrt{2\pi}R_*} \exp\left(-\frac{\nu^2}{2}\right) \times \frac{2}{(2\pi)^{N/2}} \left(\frac{\gamma}{\sqrt{N}}\right)^N \Bigg[H_{N-1}(\nu) + \sum_{n=3}^{\infty} \sum_{s=0}^{N} \gamma^{-s} \sum_{i,j=0}^{i+2j=n-s} \frac{(-N)^{j+s}(N-2)!!\, L_j^{(\frac{N-2}{2})}(0)}{i!(2j+N-2)!!} \left\langle x^i q^{2j} I_s \right\rangle_{GC} H_{i+N-s-1}(\nu) \Bigg]$$

(cf first term: Matsubara 1994-2005)

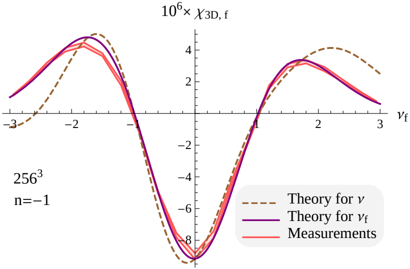
How non-Gaussianity develops, eq 2D Euler charachteristic

Excitation of Hermite modes of alternating parity



“Hermite spectroscopy”

From:



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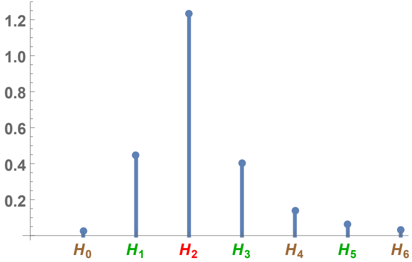


Redshift space – anisotropic statistics

Polymonial expansion for $\mathscr{P}_{3D}$ in redshift space

Symmetry:

Rotational around the line of sight (LOS along 3rd coordinate)

Variables:

linear(4) $x,\ x_3,\ x_{33},\ J_{1\perp} = x_{11} + x_{22}$

quadratic(3) $q^2 = x_1^2 + x_2^2,\ Q^2 = x_{13}^2 + x_{23}^2,\ J_{2\perp} = (x_{11} - x_{22})^2 + 4x_{12}^2$

cubic(1) $\Upsilon = \left(x_{13}^2 - x_{23}^2\right)(x_{11} - x_{22}) + 4x_{12}x_{13}x_{23}$

Gaussian limit JPDF

$$G\left(x, q_\perp^2, x_3\,\zeta, J_{2\perp}, \xi, Q^2, \Upsilon\right) = \frac{1}{4\pi^3\sqrt{Q^4 J_{2\perp} - \Upsilon^2}}\, e^{-\frac{1}{2}x^2 - q_\perp^2 - \frac{1}{2}x_3^2 - \frac{1}{2}\zeta^2 - J_{2\perp} - \frac{1}{2}\xi^2 - Q^2}$$

Uniformly distributed $\Upsilon \in [-Q^2\sqrt{J_{2\perp}}, +Q^2\sqrt{J_{2\perp}}]$ can be integrated over for Minkowski functionals, extrema ...

Euler characteristic in redshift space

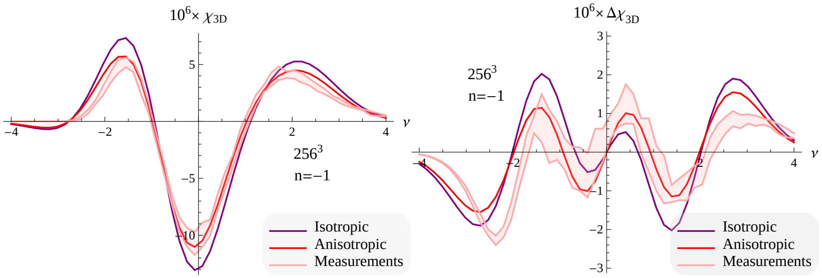
$$\chi_{2+1}(\nu) = \frac{e^{-\nu^2/2}}{8\pi^2}\left[\frac{\sigma_{1\parallel}\sigma_{1\perp}^2}{\sigma^3}H_2(\nu) + \sum_{n=3}^{\infty}\chi_{2+1}^{(n)}\right]$$

with non-Gaussian corrections $\chi_{2+1}^{(n)}$, given, to all orders, by

$$\begin{aligned}
\chi_{2+1}^{(n)}(\nu) &= \frac{\sigma_{2\perp}^2\sigma_{2\parallel}}{\sigma_{1\perp}^2\sigma_{1\parallel}}\Bigg[\sum_{\sigma_n}\frac{(-1)^{j+m}}{2^m i!\, j!\, m!}H_{i+2}(\nu)\gamma_\parallel\gamma_\perp^2\left\langle x^i q_\perp^{2j} x_3^{2m}\right\rangle_{\rm GC} \\
&- \sum_{\sigma_{n-1}}\frac{(-1)^{j+m}}{2^m i!\, j!\, m!}H_{i+1}(\nu)\left\langle x^i q_\perp^{2j} x_3^{2m}\left(\gamma_\perp^2 x_{33} + 2\gamma_\perp\gamma_\parallel J_{1\perp}\right)\right\rangle_{\rm GC} \\
&+ \sum_{\sigma_{n-2}}\frac{(-1)^{j+m}}{2^m i!\, j!\, m!}H_i(\nu)\left\langle x^i q_\perp^{2j} x_3^{2m}\left(2\gamma_\perp(J_{1\perp}x_{33} - \gamma_2 Q^2) + \gamma_\parallel(J_{1\perp}^2 - J_{2\perp})\right)\right\rangle_{GC} \\
&- \sum_{\sigma_{n-3}}\frac{(-1)^{j+m}}{2^m i!\, j!\, m!}H_{i-1}(\nu)\left\langle x^i q_\perp^{2j} x_3^{2m}\left(x_{33}(J_{1\perp}^2 - J_{2\perp}) - 2\gamma_2\left(Q^2 J_{1\perp} - \Upsilon\right)\right)\right\rangle_{GC}\Bigg]
\end{aligned}$$

Euler characteristics in redshift space

$\sigma = 0.18,\ f = 1$



$$\chi_{3D}^{0+1} = \frac{\sigma_{1\|}\sigma_{1\perp}^2}{\sigma^3}\frac{e^{-\nu^2/2}}{8\pi^2}\left[H_2(\nu) + \frac{1}{3!}H_5(\nu)\langle x^3\rangle + H_3(\nu)\left(\langle x q_\perp^2\rangle + \frac{\langle x x_3^2\rangle}{2}\right) - \frac{H_1(\nu)}{\gamma_\perp}\left(\langle J_{1\perp} q_\perp^2\rangle + \langle J_{1\perp} x_3^2\rangle\right)\right]$$

Important anisotropy measure $\beta_\sigma = 1 - \frac{\sigma_{1\perp}^2}{2\sigma_{1\|}^2} \approx \frac{4}{5} f/b$

Slicing through redshift space

$$\chi_{2\mathrm{D}}(\nu,\theta) = \frac{e^{-\nu^2/2}}{(2\pi)^{3/2}}\frac{\sigma_{1\perp}^2}{2\sigma^2}\sqrt{\frac{1-\beta_\sigma\sin^2\theta}{1-\beta_\sigma}}\times\Big[H_1(\nu)+\frac{1}{3!}H_4(\nu)\langle x^3\rangle$$
$$+H_2(\nu)\left(\langle x q_\perp^2\rangle+\frac{1}{2}\frac{\cos^2\theta}{1-\beta_\sigma\sin^2\theta}\left(\langle x x_3^2\rangle-\langle x q_\perp^2\rangle\right)\right)$$
$$-\frac{1}{\gamma_\perp}\left(\langle q_\perp^2 J_{1\perp}\rangle+\frac{1}{2}\frac{\cos^2\theta}{1-\beta_\sigma\sin^2\theta}\left(\langle x_3^2 J_{1\perp}\rangle-2\langle q_\perp^2 J_{1\perp}\rangle\right)\right)+\mathcal{O}(\sigma^2)\Big]$$

$$\mathcal{N}_2(\nu,\theta) = \frac{\sigma_{1\perp}}{2\sqrt{2}\sigma}e^{-\nu^2/2}\left(1+\beta_\sigma\frac{\cos^2\theta}{4}\right)\times\Big[1+\frac{1}{3!}H_3(\nu)\langle x^3\rangle$$
$$+\frac{1}{2}H_1(\nu)\left(\langle x q_\perp^2\rangle+\frac{1}{2}\cos^2\theta\left(1+\beta_\sigma\frac{3+5\sin^2\theta}{8}\right)\left(\langle x x_3^2\rangle-\langle x q_\perp^2\rangle\right)\right)+\mathcal{O}(\sigma^2)\Big]$$

$$\mathcal{N}_1(\nu,\theta) = \frac{\sigma_{1\perp}}{\sqrt{3}\pi\sigma}e^{-\nu^2/2}\sqrt{\frac{1-\beta_\sigma\sin^2\theta}{1-\beta_\sigma}}\times\Big[1+\frac{1}{3!}H_3(\nu)\langle x^3\rangle$$
$$+\frac{1}{2}H_1(\nu)\left(\langle x q_\perp^2\rangle+\frac{\cos^2\theta}{1-\beta_\sigma\sin^2\theta}\left(\langle x x_3^2\rangle-\langle x q_\perp^2\rangle\right)\right)+\mathcal{O}(\sigma^2)\Big]$$

Invariance of Minkowski functionals

- Minkowski functionals (but also extrema counts) are invariant under any local monotonic transformations $x = f(y)$, if one adjust threshold accordingly, $\nu_x = f(\nu_y)$. Which is achived by choosing threshold that gives the same filling factor of the volume above.
- This means that the coefficients in front of every mode in Hermite expansion are invariant separately. This can be shown perturbatively (?).
- Corollary I: combinations of cumulants in every Hermite mode are invariant wrt to any local monothonic bias. (Is it order by order in σ expansion ?)

- Corollary II: If the starting field y is Gaussian, these coefficients are zero (or sum to zero ?)
- Corollary IIa: In anisotropic case, angle dependent section of each coefficient should then vanish separately. In particular
$$\langle x q_\perp^2 \rangle = \langle x x_3^2 \rangle \;\rightarrow\; \frac{\langle x \nabla_\perp x \cdot \nabla_\perp x \rangle}{\langle x \nabla_\parallel \cdot \nabla_\parallel \rangle} = \frac{\sigma_{1\perp}^2}{\sigma_{1\parallel}^2}$$
- This holds for f_{NL} models in leading non Gaussian model explicitly (despite even transformation being local but non-monotonic in this case)
- But here we are talking about **local transformation in redshift space**.

Fiducial experiment

- Cut through redshisift space at different angles. For instance, $\theta = 0$ and $\theta = \pi/2$ and measure 2D geometrical statistics as the function of filling factor
- Do Hermite decomposition, determine H_n coefficients
- Use the ratio of main Gaussian lines to determine $\beta_\sigma = 1 - \frac{\sigma_{1\perp}^2}{2\sigma_{1\|}}$

$$\frac{H_1\left[(\chi_{2D}(\nu_f, \pi/2)\right]}{H_1\left[\chi_{2D}(\nu_f, 0)\right]} = \sqrt{1-\beta_\sigma} + \mathscr{O}(\sigma^2) \qquad \frac{H_0\left[\mathscr{N}_2(\nu_f, \pi/2)\right]}{H_0\left[\mathscr{N}_2(\nu_f, 0)\right]} = \sqrt{1-\beta_\sigma} + \mathscr{O}(\sigma^2)$$

  Different statistics have different corrections $\mathscr{O}(\sigma^2)$, which can be leveraged for control

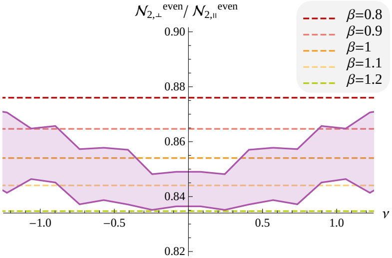
- Use angle variable ratio of a secondary to Gaussian line and β_σ to determine normalized cumulant combinations, and their anisotropy

$$\frac{H_2\left[\mathscr{N}_2(\nu_f, \theta)\right]}{H_0\left[\mathscr{N}_2(\nu_f, \theta)\right]} \rightarrow \quad \langle x q_\perp^2 \rangle \; , \; \langle x q_\perp^2 \rangle - \langle x x_3^2 \rangle$$

- Using theoretical insight (PT) relate β_σ and higher order redshift cumulants to real space σ, f, b .... To evaluate σ 3D statistics have most power.

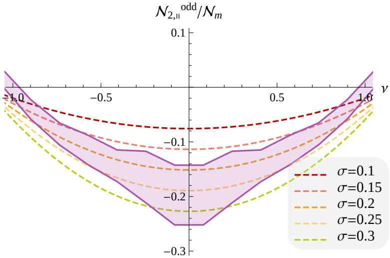
Using non-Gaussianity and redshift distortions of geometrical measures of the Cosmic Web to recover

$\beta = \Omega^{0.55}/b$



Reconstructed β from angular dependence of Minkowski functionals in 2D slices

σ (and D(z))



Reconstructed σ from non-Gaussian corrections of Minkowski functionals

Conclusions

- Redshift space distortions are not just nuisance, but a source of additional information about cosmology of the Universe.
- Redshift space analysis is in principle capable of mining more information than real space analysis. For that one needs to leverage the anisotropic properties of redshift space.
- Generalizing previous work, we have developed complete theory of Minkowski functionals in the bulk and on slices of anisotropic and mildly non-Gaussian fields.
- At mildly non-linear scales of cosmological structure formation, non-Gaussian and redshift effects can be, to some extend, disentangled.