Statistics of Merging Peaks of Random Gaussian Fluctuations: Skeleton Tree Formalism

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Large scale filamentary structure connecting clusters of galaxies is evident both in data and simulations.

Context: geometry & Topology

Sloan DS Survey

Skeleton & Walls of Horizon-AGN

Geometry of LSS as a probe of cosmology
Context: feeding galaxies via cold flows?

Large scale filamentary structure connecting clusters of galaxies is evident both in data and simulations.

Filamentary accretion regulating stellar formation / AGN feedback/AM acquisition
The skeleton tree formalism

Can we build a merger-tree like structure from the initial conditions?

⇒ Yes! Study the topological structure of the ICs at different scales (Hanami 2001)

Extend Hanami '01 to other critical events
The skeleton tree formalism

Can we build a merger-tree like structure from the initial conditions?

⇒ Yes! Study the topological structure of the ICs at different scales (Hanami 2001)

Context: skeleton tree

Extend Hanami '01 to other critical events

Marulli+2009

Cadiou+ in prep

Thursday, 11 April, 19
Spherical collapse: time-smoothing duality
Spherical collapse: time-smoothing duality

Context

Understand special events in evolution of cosmic web
Change in excursion's topology impacts galaxy formation
1D outlook: ridges in position-smoothing landscape

≠ smoothing

density

position

smoothings

position
1D outlook

Special event: topology of underlying field changes
Building the skeleton tree
3D outlook

- GRF smoothed at ≠ scales
- identify critical points
- build skeleton tree
- find critical events
3D outlook
\[ \frac{\partial^2 N}{\partial r^3 \partial R} \equiv \langle \delta^{(3)}_\text{D}(\mathbf{r} - \mathbf{r}_0) \delta_\text{D}(R - R_0) \rangle , \]

where \( \mathbf{r}_0 \) is a (double) critical point in real space and \( R_0 \) the scale at which the two critical points merge.

\[ \checkmark \text{Invoke ergodicity} \]
\[ \checkmark \text{Change variable to (gradient, determinant)} \]

At \((R_0, \mathbf{r}_0)\) gradient and one eigenvalue vanish
\[
\frac{\partial^2 \mathcal{N}}{\partial r^3 \partial R} \equiv \langle \delta_D^{(3)}(\mathbf{r} - \mathbf{r}_0)\delta_D(R - R_0) \rangle,
\]
where \(\mathbf{r}_0\) is a (double) critical point in real space and \(R_0\) the scale at which the two critical points merge.

\[
d(\delta) \equiv \det(\nabla \nabla \delta) = \lambda_1 \lambda_2 \lambda_3
\]

\[
\frac{\partial^2 \mathcal{N}}{\partial r^3 \partial R} = \left\langle J \delta_D^{(3)}(\nabla \delta)\delta_D(d) \right\rangle
\]

\[
J(d, \delta) = \begin{vmatrix}
\partial_R d & \tilde{\nabla} d \\
\partial_R \tilde{\nabla} \delta^T & \tilde{\nabla} \tilde{\nabla} \delta
\end{vmatrix}
\]
Let us therefore first define the determinant of the hessian
\[ \frac{\partial^2 N}{\partial r^3 \partial R} \equiv \langle \delta^{(3)}_D (r - r_0) \delta_D (R - R_0) \rangle, \]
where \( r_0 \) is a (double) critical point in real space and \( R_0 \) the scale at which the two critical points merge.

\[ d(\delta) \equiv \det(\nabla \nabla \delta) = \lambda_1 \lambda_2 \lambda_3 \]

\[ \frac{\partial^2 N}{\partial r^3 \partial R} = \langle J \, \delta^{(3)}_D (\nabla \delta) \delta_D (d) \rangle \]

\[ J(d, \delta) = \begin{vmatrix} \partial_R d & \nabla d & = & \partial_R d & \nabla d \\ \partial_R \nabla \delta^T & \nabla \nabla \delta & \end{vmatrix} = \begin{vmatrix} \partial_R d & \nabla d \\ -R \nabla \Delta \delta^T & \nabla \nabla \delta \end{vmatrix}, \]

for a Gaussian filter
\[ \partial_R \delta = -R \Delta \delta \]
\[
\frac{\partial^2 N}{\partial r^3 \partial R} \equiv \langle \delta_D^{(3)}(r - r_0) \rangle
\]

where \(r_0\) is a (double) critical point, at which the two critical points merge to produce a critical ridge. For example a maximum and a saddle:

\[
d(\delta) \equiv \text{det}(\nabla \nabla \delta)
\]

\[
\frac{\partial^2 N}{\partial r^3 \partial R} = \left\langle J(\delta_D^{(3)}(\nabla \delta)) \delta_D(d) \right\rangle
\]

\[
J(d, \delta) = \begin{vmatrix}
\partial_R d & \nabla d \\
\partial_R \nabla \delta^T & \nabla \nabla \delta
\end{vmatrix}
\]

\[
\nabla d = \begin{vmatrix}
\partial_R d & \nabla d \\
- R \nabla \Delta \delta^T & \nabla \nabla \delta
\end{vmatrix}
\]

for a Gaussian filter

\[
\partial_R \delta = - R \Delta \delta
\]
Derivation

\[
\frac{\partial^2 n}{\partial R \partial \nu} = \frac{\partial^5 N}{\partial r^3 \partial R \partial \nu},
\]

\[
= \frac{1}{R} \frac{\left| x_{11} x_{22} \right| \left| x_{3ii} \right| \left| x_{333} \right| \delta_D^{(3)}(x_i) \delta_D(x_{33}) \delta_D(x - \nu)}{R^3 \tilde{R}},
\]

\[
= \frac{10^3 x R^4 \partial^4 N \partial r^3 \partial R}{\text{cf skeleton length: packing of transverse curvature + zero longitudinal curvature}}.
\]
\[
\frac{\partial^2 n}{\partial R \partial \nu} = \frac{\partial^5 N}{\partial r^3 \partial R \partial \nu},
\]
\[
= R \frac{\left\langle |x_{11} x_{22}| |x_{3ii}||x_{333}| \delta_D^{(3)}(x_i) \delta_D(x_{33}) \delta_D(x - \nu) \right\rangle}{R^3 \tilde{R}},
\]

E.g. number density of peak-filament mergers to the number density of filament-wall mergers

\[
r_{2/1} = \frac{24 \sqrt{3}}{29 \sqrt{2} - 12 \sqrt{3}} \approx 2.05508.
\]
Derivation

\[ \frac{\partial^2 n}{\partial R \partial \nu} = \frac{\partial^5 N}{\partial r^3 \partial R \partial \nu}, \]

\[ = R \left\langle \left| x_{11} x_{22} \right| \begin{vmatrix} \left| x_{3ii} \right| \end{vmatrix} \left| x_{333} \right| \delta_D^{(3)}(x_i) \delta_D(x_3) \delta_D(x - \nu) \right\rangle R^3 \tilde{R}, \]

E.g. number density of peak-filament mergers to the number density of filament-wall mergers

\[ r_{2/1} = \frac{24 \sqrt{3}}{29 \sqrt{2} - 12 \sqrt{3}} \approx 2.05508. \]

10^3 x R^4 \frac{\partial^4 N}{\partial r^3 \partial R}

cf. skeleton length: packing of transverse curvature + zero longitudinal curvature

\[
\frac{1}{8 \sqrt{5} \pi^{3/2} (6 - 5 \gamma^2)^4 (5 \gamma^2 - 9)^2} \left( -8 \sqrt{\pi} (6 - 5 \gamma^2)^{7/2} (5 \gamma^2 - 9)^3 e^{2 \gamma^2} \right) \left( 2 \sqrt{2} (6 - 5 \gamma^2)^4 \sqrt{9 - 5 \gamma^2} e^{\gamma^2} \right)^2 \left( 2 \sqrt{2} (6 - 5 \gamma^2)^4 \sqrt{9 - 5 \gamma^2} e^{\gamma^2} \right)^2 \left[ 3600 \gamma^4 \gamma^6 + 120 \gamma^2 (5 \gamma^2 - 9) (35 \gamma^2 - 27) \gamma^2 + (9 - 5 \gamma^2)^2 (575 \gamma^4 - 1230 \gamma^2 + 783) \right] - 8 \sqrt{\pi} (6 - 5 \gamma^2)^{7/2} (5 \gamma^2 - 9)^3 e^{2 \gamma^2} \left( 2 \sqrt{2} (6 - 5 \gamma^2)^4 \sqrt{9 - 5 \gamma^2} e^{\gamma^2} \right)^2 \left( 2 \sqrt{2} (6 - 5 \gamma^2)^4 \sqrt{9 - 5 \gamma^2} e^{\gamma^2} \right)^2 \left( 3600 \gamma^4 \gamma^6 + 120 \gamma^2 (5 \gamma^2 - 9) (35 \gamma^2 - 27) \gamma^2 + (9 - 5 \gamma^2)^2 (575 \gamma^4 - 1230 \gamma^2 + 783) \right) \frac{\gamma^2}{\sqrt{5} \gamma^2 - 14 \gamma^2} + 6 \right) -
\]
In (invariant) Hessian frame:

**extrema**

\[ n_{\text{ext}} = \int P(x) \prod_{1 \leq i \leq 3} \delta_D(x_i) \lambda_i \, dx \]

volume \( \propto 1/\lambda_1 \lambda_2 \lambda_3 \)

packing sphere problem: curvature.

**BBKS**

**skeleton**

\[ n_{\text{skl}} = \int P(x) \prod_{1 < i \leq 3} \delta_D(x_i) \lambda_i \, dx \]

section \( \propto 1/\lambda_2 \lambda_3 \)

packing tube problem: transverse curvature.

**SPCP**
If the field is Gaussian (large scales/early times), the total number density of critical points then reads

\[
\begin{align*}
\langle n_{\text{max}} \rangle &= \langle n_{\text{min}} \rangle = \frac{1}{8\sqrt{3}\pi R_*^2} \\
\langle n_{\text{sad}} \rangle &= \frac{1}{4\sqrt{3}\pi R_*^2}
\end{align*}
\]

And as a function of peak height (analytical in 2D, not in 3D):

\[
\begin{align*}
\langle n_{\text{max}} \rangle &= \langle n_{\text{min}} \rangle = \frac{29\sqrt{15} - 18\sqrt{10}}{1800\pi^2 R_*^3} \\
\langle n_{\text{sadf}} \rangle &= \langle n_{\text{sadw}} \rangle = \frac{29\sqrt{15} + 18\sqrt{10}}{1800\pi^2 R_*^3}
\end{align*}
\]

\[R_* = \sigma_1 / \sigma_2 = \text{distance between peaks}\]
Map event count to \((z,M)\)

\[
\frac{\partial^2 n}{\partial \log M \partial z} \bigg|_c = \frac{\partial^2 n}{\partial R \partial \nu} \bigg|_c \delta_c \frac{dD}{dz} \frac{R}{3}
\]

where \(\nu \sigma_0 = \delta_c D(z)\)

**Work in progress**
On the connectivity of halos

Compute frequency of filament merger compared to halo merger in the vicinity of a halo merger event $\xi_{hf}(r)/\xi_{hh}(r)$.

$$1 + \xi_p = \frac{\langle \text{cond}_p(x) \text{cond}_p(y) \rangle}{\langle \text{cond}_p(x) \rangle^2}, \quad 1 + \xi_f = \frac{\langle \text{cond}_f(x) \text{cond}_p(y) \rangle}{\langle \text{cond}_f(x) \rangle \langle \text{cond}_p(x) \rangle}$$
Global connectivity for GRF

How many filaments connect to a node?

Number of connected saddles are measured using the *Disperse* skeleton algorithm (Sousbie +11) in GRF realisations.

Can we predict the mean connectivity?
Because each filament goes through one and only one saddle pt, on average:

\[
\langle \kappa \rangle = \frac{2 \bar{n}_{sad}}{\bar{n}_{\text{max}}}
\]

\[
= 4 \quad \text{in 2D GRF}
\]

\[
= \frac{2 \left( 1057 + 348 \sqrt{6} \right)}{625} \approx 6.11 \quad \text{in 3D GRF}
\]
Because each filament goes through one and only one saddle pt, on average:

\[
\langle \kappa \rangle = \frac{2\bar{n}_{sad}}{\bar{n}_{max}} = 4 \quad \text{in 2D GRF}
\]

\[
= \frac{2 \left( 1057 + 348\sqrt{6} \right)}{625} \approx 6.11 \quad \text{in 3D GRF}
\]

In d dimensions, (relying on numerical integrations):

\[
\kappa = 2d + \left( \frac{2d - 4}{7} \right)^{7/4}
\]
2D "ideal" cosmic environment:

Mean local cosmic initial condition **homeomorphic** to such crystal.
3D "ideal" cosmic environment

Mean local cosmic field quasi homeomorphic to such crystal

3D connectivity: topology

tube saddle

wall saddle

maximum

minimum
GRF connectivity PDF: dependence with scale $n_s$

Full distribution of connectivity:

Weak dependency on $n_s$
GRF connectivity PDF: dependence with scale/n_s

Full distribution of connectivity:

\[ \langle \kappa \rangle = \frac{2 \left( 1057 + 348\sqrt{6} \right)}{625} \]

no \( R_* \) dependency!!

- \( n_s = 0 \)
- \( n_s = -1 \)
- \( n_s = -2 \)
- \( n_s = -3 \)

Weak dependency on \( n_s \)
GRF connectivity: dependence with peak height

Dependence with peak height:

\[ \nu = \frac{\delta}{\sigma} \]

The rarer the peak, the more connected
GRF connectivity: dependence with peak height

Joint PDF of $\kappa$ and $\nu$ in 3D

Notable Result:

- High peaks tend to have more connections
- Peaks with large number of connections are predominantly high
- mean $\langle \kappa | \nu \rangle$, 6.5 ($\nu = 2$), 10 ($\nu = 3$)
Connectivity as a function of Mass in Horizon-4π: 1 000 000 halos.
Connectivity as a function of Mass in Horizon-4π: 1 000 000 halos.
Analogy with sphere packing pb
Same ideas can be used to also predict the **clustering of peaks** by means of their 2 point correlation function (also applies to peak saddle etc.):

\[
1 + \xi_{pp}(r, \nu) = \frac{\left< \rho_{pk}(X) \rho_{pk}(Y) \right>}{\left< \rho_{pk}(X) \right>^2}.
\]

where the localized peak number density \( \rho_{pk}(X) \),

\[
\rho_{pk}(X) = \frac{1}{R^d} |\det(x_{ij})| \delta_D(x_i) \Theta_H(-\lambda_i) \delta_D(x - \nu),
\]

exclusion is essential to understand connectivity
Towards connectivity theory

Idea: Count the number of saddles up to \( R_{\text{max}} \ldots \), conditional on the properties of the peak. But what is \( R_{\text{max}} \)? Some characteristic size of a peak-patch around the peak. Let us look (in 3D) where the neighbouring peaks are using peak-peak correlation function

They are at the end of the exclusion zone, which for high central peak \( \nu \geq 2 \) it increases with \( \nu \) roughly linearly

\[
R_{\text{max}} \approx (0.9 + \nu/5)R_p \quad R_{\text{max}} = 1.2, 1.5, 1.8 \quad R_p, \quad \nu = 2, 3, 4
\]
Global connectivity for GRF: theory

Estimating $\kappa$ by counting saddles to the next peak

From the Peak-Saddle correlation function

$$\kappa(\nu) = \bar{n}_{\text{sad}} \int_0^{R_{pp}} d^D r (1 + \xi_{\text{pk-sad}}(r, \nu))$$

Number of saddles to distance $R_{\text{max}}$ conditional on the height $\nu$ of the peak translates to peak connectivity $\langle \kappa | \nu \rangle$.

Subtle interplay between clustering of saddles and zone of influence of peak.
Connectivity of the Cosmic Web

Connectivity of a non-Gaussian field differ from the Gaussian

- In cosmological simulations, as density becomes more non-Gaussian, connectivity of the Cosmic Web decreases.
- This leads to model dependent history of the connectivity at different redshifts.

\[ P(\kappa) \]

\[ \langle \kappa \rangle \]

Filaments merge in a cosmology-dependent way!
Non-Gaussian 3D Extrema Counts (Gay et al, 2011)

\[
\langle n_\pm \rangle = \frac{29\sqrt{15} \mp 18\sqrt{10}}{1800\pi^2 R_*^3} + \frac{5\sqrt{5}}{24\pi^2 \sqrt{6\pi} R_*^3} \left( \langle q^2 J_1 \rangle - \frac{8}{21} \langle J_1^3 \rangle + \frac{10}{21} \langle J_1 J_2 \rangle \right)
\]

With Gram Charlier expansion, prediction at arbitrary order

At \( \sigma \approx 0.2 \)

\[
\frac{\langle n_{\text{saddle}} \rangle}{\langle n_{\text{peak}} \rangle} \approx 2.8 \quad \Rightarrow \langle \kappa \rangle \approx 5.6
\]
Local multiplicity and bifurcation points

For galaxy formation, what matters most is how many filament connect locally onto a galaxy. At small enough scale, a peak is always ellipsoidal so that only two branches of filament stick out. Then those branches bifurcate. Some bifurcations appear so close to the peak that they are physically irrelevant. Hence we will define the multiplicity as the local number of filaments.
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\[ \mu = \kappa - n_{\text{bifurcations}} \]

- \( \mu \approx 3 \) in 2D
- \( \mu \approx 4 \) in 3D

![Graph showing the distribution of multiplicity \( \mu \) with different values of \( n_s \)]

For 2D, \( \langle \mu \rangle = 3 \).
Local multiplicity and bifurcation points

For galaxy formation, what matters most is how many filament connect **locally** onto a galaxy. At small enough scale, a peak is always **ellipsoidal** so that only two branches of filament stick out. Then those branches **bifurcate**. Some bifurcations appear so close to the peak that they are physically irrelevant. Hence we will define the **multiplicity** as the local number of filaments.

\[
\mu = \kappa - n_{\text{bifurcations}}
\]

\[\mu \approx 3 \quad \text{in 2D}\]

\[\mu \approx 4 \quad \text{in 3D}\]

3D: \(\langle \mu \rangle = 4\)
Local multiplicity

The denser the environment, the higher the multiplicity (e.g. bringing less coherent angular momentum and generating more ellipsoidal galaxies)
Let us count filament crossings at a **sphere** of radius $R$ around the central peak...
Local multiplicity: towards a theoretical prediction

Size of peak patches depends on their height
Local multiplicity: towards a theoretical prediction

Not all filaments are equally prominent. Counting important ones

- Number of dense $\nu_f > 2$ filamentary bridges is increasing with the height of the central peak
- Not very rare $\nu = 3$ central peak has two (branches of) dense filaments, i.e it sits in one dominant filament on average
- Rare $\nu = 4$ peak is at intersection of three prominent branches.
Typically, two to three dense filaments dominate and therefore define a plane of accretion... in agreement with numerical simulation (Danovich+12) and observations of plane of satellites around galaxies.
On the connectivity of halos

Compute frequency of filament merger compared to halo merger in the vicinity of a halo merger event, \( \xi_{hf}(r)/\xi_{hh}(r) \).

\[
1 + \xi_p = \frac{\langle \text{cond}_p(x)\text{cond}_p(y) \rangle}{\langle \text{cond}_p(x) \rangle^2}, \quad 1 + \xi_f = \frac{\langle \text{cond}_f(x)\text{cond}_p(y) \rangle}{\langle \text{cond}_f(x) \rangle \langle \text{cond}_p(x) \rangle}
\]
Connectivity is a packaging pb because of exclusion

- Connectivity = number of filament connected
  - $\kappa = 4$ in 2D  $\kappa = 6.11$ in 3D  (for GRF)
- Multiplicity = number of local filament connected
  - $\mu \sim 3$ in 2D  $\mu \sim 4$ in 3D
- Both can be predicted from first principle
- Hence useful for cosmology & galaxy formation
Set of critical events = useful topological compression of ICs
  • impacts ‘dressed’ mergers: ML on morphology?
    (i.e. cosmic evolution of peaks and their filaments and walls).

Clustering of filament disappearance is consistent
  with *preserving connectivity* of peaks as they merge:
    • the rarer the peak the higher the rate of filaments merging.

Rate of wall disappearance = dark energy probe,
  depend on the growth rate of structure and $\sigma 2/\sigma 1\sigma 3$. 
Peak and constrained random field theories are paramount to understand the birth and growth of the cosmic web.

Many analytical results can be obtained in the weakly non-linear regime.

The topology and geometry of the cosmic web carries important cosmological information and is key for galaxy evolution.

In particular, we now have a precise understanding of the connectivity of the cosmic web (the cosmic crystal) and its evolution through statistics of critical events.

IMHO of interest beyond cosmology
Application: impact of AGN feedback?

X-Ray detected groups

filaments from galaxy distribution

Elise Darragh-Ford
Laigle, Gozaliasl, Pichon, Devriendt, Slyz et al.

Galaxy distribution, gas density — Horizon-AGN simulation (Dubois+14)
Filament extraction in 2D around groups

COSMOS reference photometric field
1 million of galaxies

Redshift and mass range constrained by galaxy photometric properties:
We work in $0.5 < z < 1.2$ with all galaxies more massive than $10^{10}$ solar mass

Darragh-Ford, Laigle et al. in prep

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Redshift and mass range constrained by galaxy photometric properties: We work in $0.5 < z < 1.2$ with all galaxies more massive than $10^{10}$ solar mass
Group Multiplicity
Measuring connectivity with photometric filaments

- Connectivity 1
- Connectivity 2
- Connectivity 3
- Connectivity 4

BGG: brightest group galaxy

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Group Multiplicity in Horizon-AGN
Testing the impact of photometric uncertainties

Virtual galaxies

Virtual (Lensed) Image
Group Multiplicity in Horizon-AGN
Testing the impact of photometric uncertainties

Hydrodynamical simulation Horizon-AGN
Dubois+14

Mock observation generation (photometry, photo-z, photometric masses)
Adding errors (including systematics)

3D reference skeleton

Mock image
Virtually observed skeleton

Confrontation

RA
DEC

-1 0 1
-1 0 1
photo-z uncertainties decrease connectivity

Darragh-Ford, Laigle et al. in prep
Group Multiplicity
Impact of connectivity on group properties

Mean connectivity increases with halo/BGG mass

see also
Theoretical predictions from Codis et al. 2018
The impact of **Multiplicity** on BGG properties

**Interpretation from Horizon-AGN simulation**

Darragh-Ford, Laigle et al. in prep
At a given halo mass, “AGN quenching efficiency” is higher at higher connectivity.

HORIZON-AGN simulation result:

Impact of connectivity on group properties

- Connectivity: proxy for mass of accreted matter; more accretion → higher feedback?
- Higher connectivity → accretion more isotropic
At a given halo mass, “AGN quenching efficiency” is higher at higher connectivity.

**HORIZON-AGN simulation result:**
At a given halo mass, “AGN quenching efficiency” is higher at higher connectivity.

- Connectivity: proxy for mass of accreted matter; more accretion $\rightarrow$ higher feedback?
- Higher connectivity $\rightarrow$ accretion more isotropic

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**Darragh-Ford, Laigle, et al in prep**

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**Impact of connectivity on group properties**

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**Group Multiplicity**

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**Thursday, 11 April, 19**
Generalized geometrical $S_n$

**Purpose:** Express the invariant **cumulants** in terms of $\sigma$ (hence $D(z)$) through Perturbation theory *e.g.* $\langle J_1 x \rangle = \text{function}(\sigma)$

\[
F_2(k_1, k_2) = \frac{5}{7} + \frac{k_1 \cdot k_2}{k_1^2} + \frac{2(k_1 \cdot k_2)^2}{7k_1^2k_2^2} \implies F_{\alpha, \beta, \gamma}(k_1, k_2) = F_2(k_1, k_2) G_{\alpha, \beta, \gamma}(k_1, k_2)
\]

**GRAVITY**

Geometric shape factor = powers of $k$

\[
\delta^{(n)}(\vec{k}) = \int d^3 \vec{k}_1 \cdots d^3 \vec{k}_n \delta^D(\vec{k} - (\vec{k}_1 + \cdots + \vec{k}_n)) F_n(\vec{k}_1, \ldots, \vec{k}_n) \delta^{(1)}(\vec{k}_1) \cdots \delta^{(1)}(\vec{k}_n),
\]
**Generalized geometrical $S_n$**

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F_2(k_1, k_2) = \frac{5}{7} + \frac{k_1 \cdot k_2}{k_1^2} + \frac{2}{7} \frac{(k_1 \cdot k_2)^2}{k_1^2 k_2^2} \implies F_{\alpha,\beta,\gamma}(k_1, k_2) = F_2(k_1, k_2)G_{\alpha,\beta,\gamma}(k_1, k_2)
\]

**GRAVITY**

**Geometric shape factor** = powers of $k$

\[
\frac{1}{\sigma} \langle x^3 \rangle = 3 \; _2\!F_1 \left( \frac{3 + n}{2}, \frac{3 + n}{2}, \frac{3}{2}, \frac{1}{4} \right) - \frac{1}{7} (8 + 7n) \; _2\!F_1 \left( \frac{3 + n}{2}, \frac{3 + n}{2}, \frac{5}{2}, \frac{1}{4} \right).
\]

**Power spectrum index**

**Skewness of field**

Lokas 94
Generalized geometrical $S_n$

**Purpose:** Express the invariant **cumulants** in terms of $\sigma$ (hence $D(z)$) through Perturbation theory e.g. $\langle J_1 x \rangle = \text{function}(\sigma)$

$$F_2(k_1, k_2) = \frac{5}{7} + \frac{k_1 \cdot k_2}{k_1^2} + \frac{2}{7} \frac{(k_1 \cdot k_2)^2}{k_1^2 k_2^2} \implies F_{\alpha,\beta,\gamma}(k_1, k_2) = F_2(k_1, k_2) G_{\alpha,\beta,\gamma}(k_1, k_2)$$

**GRAVITY**

Geometric shape factor = powers of $k$

\[
\frac{1}{\sigma} \langle x^3 \rangle = 3 \, _2 F_1 \left( \frac{3 + n}{2}, \frac{3 + n}{2}, \frac{3}{2}, \frac{1}{4} \right) - \frac{1}{7} (8 + 7n) \, _2 F_1 \left( \frac{3 + n}{2}, \frac{3 + n}{2}, \frac{5}{2}, \frac{1}{4} \right). 
\]

**Power spectrum index**

**Skewness of field**

\[
\frac{1}{\sigma} \langle x x^2 \rangle = \frac{4(48 + 62n + 21n^2)}{21n^2} \, _2 F_1 \left( \frac{3 + n}{2}, \frac{3 + n}{2}, \frac{3}{2}, \frac{1}{4} \right) - \frac{6(3 + n)(8 + 7n)}{21n^2} \, _2 F_1 \left( \frac{3 + n}{2}, \frac{5 + n}{2}, \frac{3}{2}, \frac{1}{4} \right). 
\]

3pt field- gradient cumulant

\[
n = -3 : \quad \frac{1}{\sigma} \langle x^3 \rangle = \frac{34}{7} \implies \frac{1}{\sigma} \langle x x^2 \rangle = \frac{34}{7} \frac{2}{3^2} 
\]

Lokas 94
Generalized geometrical $S_n$

**Purpose:** Express the invariant *cumulants* in terms of $\sigma$ (hence $D(z)$) through Perturbation theory e.g. \( \langle J_1 x \rangle = \text{function}(\sigma) \)

\[
F_2(k_1, k_2) = \frac{5}{7} + \frac{k_1 \cdot k_2}{k_1^2} + \frac{2}{7} \frac{(k_1 \cdot k_2)^2}{k_1^2 k_2^2} \quad \Rightarrow \quad F_{\alpha, \beta, \gamma}(k_1, k_2) = F_2(k_1, k_2) G_{\alpha, \beta, \gamma}(k_1, k_2)
\]

**GRAVITY**

Geometric shape factor= powers of $k$

\[
\frac{1}{\sigma} \langle x^3 \rangle = 3 \ {}_2 F_1 \left( \frac{3+n}{2}, \frac{3+n}{2}, \frac{3}{2}, \frac{1}{4} \right) - \frac{1}{7} (8+7n) \ {}_2 F_1 \left( \frac{3+n}{2}, \frac{3+n}{2}, \frac{5}{2}, \frac{1}{4} \right).
\]

**skewness of field**

\[
\frac{1}{\sigma} \langle x x_1^2 \rangle = 4 \frac{(48 + 62n + 21n^2)}{21n^2} \ {}_2 F_1 \left( \frac{3+n}{2}, \frac{3+n}{2}, \frac{3}{2}, \frac{1}{4} \right) - \frac{6(3+n)(8+7n)}{21n^2} \ {}_2 F_1 \left( \frac{3+n}{2}, \frac{5+n}{2}, \frac{3}{2}, \frac{1}{4} \right).
\]

**3pt field- gradient cumulant**

\[
n = -3 : \quad \frac{1}{\sigma} \langle x^3 \rangle = \frac{34}{7} \quad \Rightarrow \quad \frac{1}{\sigma} \langle x x_1^2 \rangle = \frac{34}{7} \frac{2}{3^2}
\]

See also Bouchet for S4

Lokas 94
**Generalized geometrical $S_n$**

**Purpose:** Express the invariant **cumulants** in terms of $\sigma$ (hence $D(z)$) through Perturbation theory *e.g.* $\langle J_1 x \rangle = \text{function}(\sigma)$

$F_2(k_1, k_2) = \frac{5}{7} + \frac{k_1 \cdot k_2}{k_1^2} + 2 \sum_{r=1}^{\infty} R_r F_{2r}(k_1, k_2) = F_2(k_1, k_2)G_{\alpha, \beta, \gamma}(k_1, k_2)$

**GRAVITY**

Generalized geometrical cumulants in powers of $k$

$\langle x^3/\sigma \rangle = 3.144 \pm 0.08$

$\langle x^2J_1/\sigma \rangle = -3.248 \pm 0.06$

$\langle xJ_1^2/\sigma \rangle = 3.871 \pm 0.06$

$\langle xJ_2/\sigma \rangle = 1.545 \pm 0.02$

$\langle q^2J_1/\sigma \rangle = -1.335 \pm 0.02$

$\langle J_1^3/\sigma \rangle = -4.644 \pm 0.08$

$\langle J_1J_2/\sigma \rangle = -0.679 \pm 0.01$

$\langle J_3/\sigma \rangle = 1.304 \pm 0.03$

1. **Power spectrum index**

$$\frac{1}{\sigma} \langle x_1^3 \rangle = 3 \cdot 2F_1 \left( \frac{3+n}{2}, \frac{3+n}{2}, \frac{3}{2}, \frac{1}{4} \right) - \frac{1}{7} \left( 8 + 7n \right) 2F_1 \left( \frac{3+n}{2}, \frac{3+n}{2}, \frac{5}{2}, \frac{1}{4} \right).$$

2. **Skewness of field**

$$\frac{1}{\sigma} \langle xx_1^2 \rangle = \frac{4(48 + 62n + 21n^2)}{21n^2} 2F_1 \left( \frac{3+n}{2}, \frac{3+n}{2}, \frac{3}{2}, \frac{1}{4} \right) - \frac{6(3+n)(8 + 7n)}{21n^2} 2F_1 \left( \frac{3+n}{2}, \frac{5+n}{2}, \frac{3}{2}, \frac{1}{4} \right).$$

3. **3pt field- gradient cumulant**

$$n = -3 : \quad \frac{1}{\sigma} \langle x^3 \rangle = \frac{34}{7} \quad \Rightarrow \quad \frac{1}{\sigma} \langle xx_1^2 \rangle = \frac{34}{7} \cdot \frac{2}{3^2}.$$
Fiducial DE experiment

- Generate scale invariant ICs
- Evolve them with gravity
- Identify critical sets
- Compute differential counts
- Estimate amplitude of NG distortion via PT
- Deduce geometric critical set $\sigma$

**filament saddle points**

**Unbiased geometric Dark Energy estimate up to $\sigma \sim 0.2$**
How is the cosmic web woven?

- Context
- Random fields, Peak theory, critical events
- Cosmic connectivity
- Cosmic multiplicity
- Application: AGN in groups
\[
\frac{\partial^2 n}{\partial R \partial \nu}\bigg|_\pm = \frac{3\sqrt{3}(1-\tilde{\gamma}^2)(25\gamma^4 + 30\gamma^2(2\nu^2 - 1) - 27)R}{20\sqrt{10\pi^{5/2}}(9 - 5\gamma^2)^{5/2}R_*^3 \tilde{R}^2} e^{-\frac{9\nu^2}{2(9 - 5\gamma^2)}}.
\]