Higher-order corrections to the peak clustering

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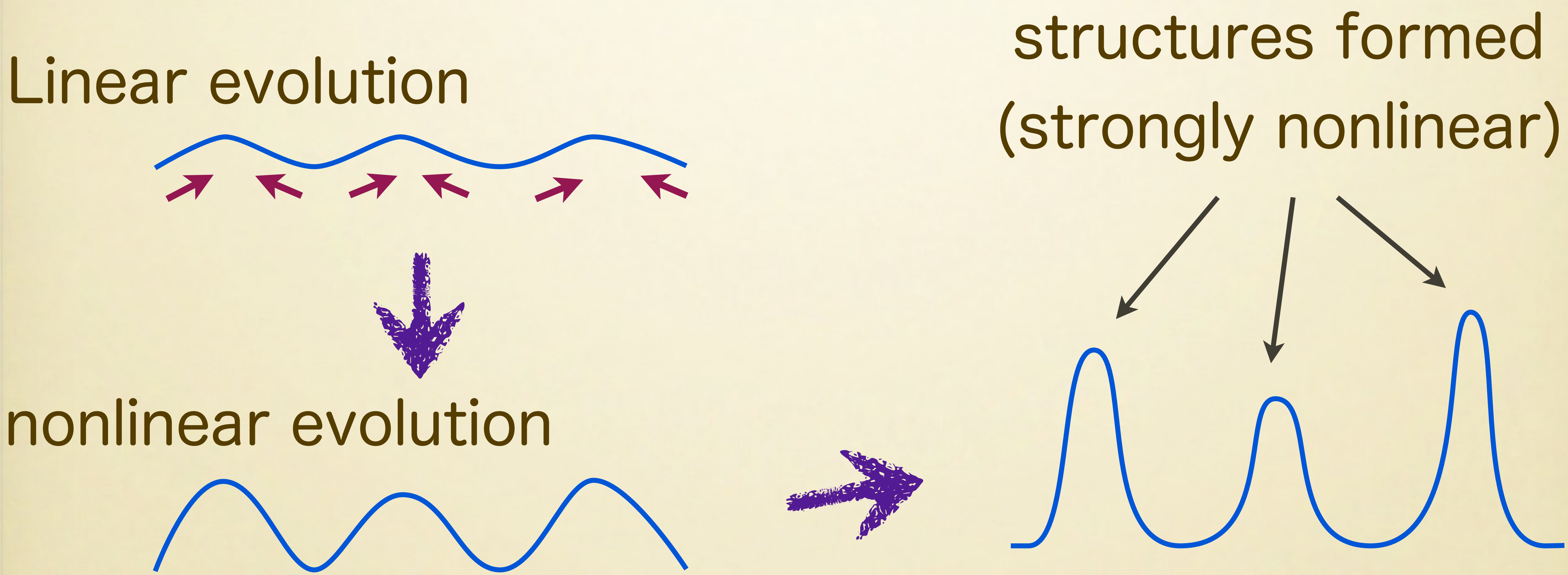
Introduction

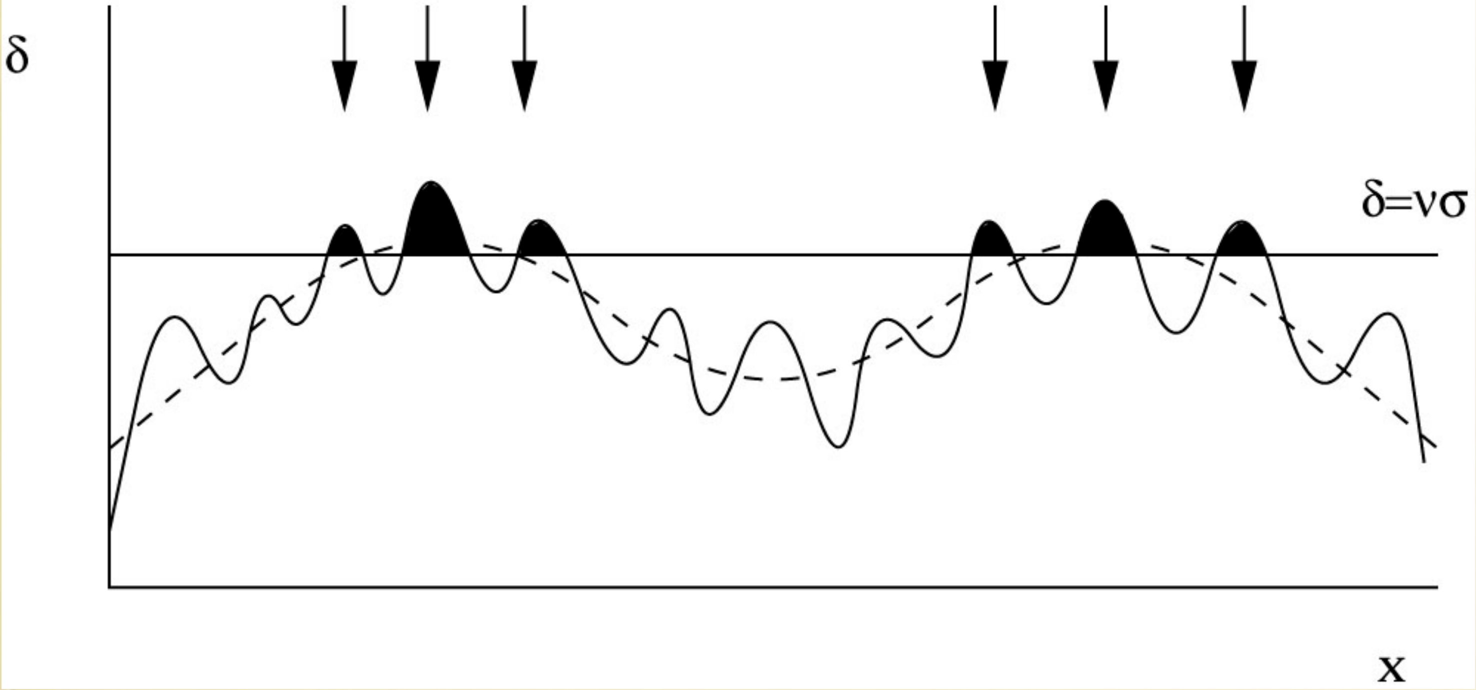
Motivation 1

- Density peaks: formation sites of the structure formation in the universe



Motivation 2

- Clustering properties of primordial black holes (PBH)



Radiation dominated phase:

Jeans length ~ Horizon radius
~ Schwarzschild radius of horizon mass
=> PBH

- So far, PBH abundances are much discussed, but little is known about clustering properties
  - clustering of peaks ?

Initial clustering of PBH?

Chisholm (2006)

Ali-Haimoud (2018),
Desjacques and Riotto (2018)

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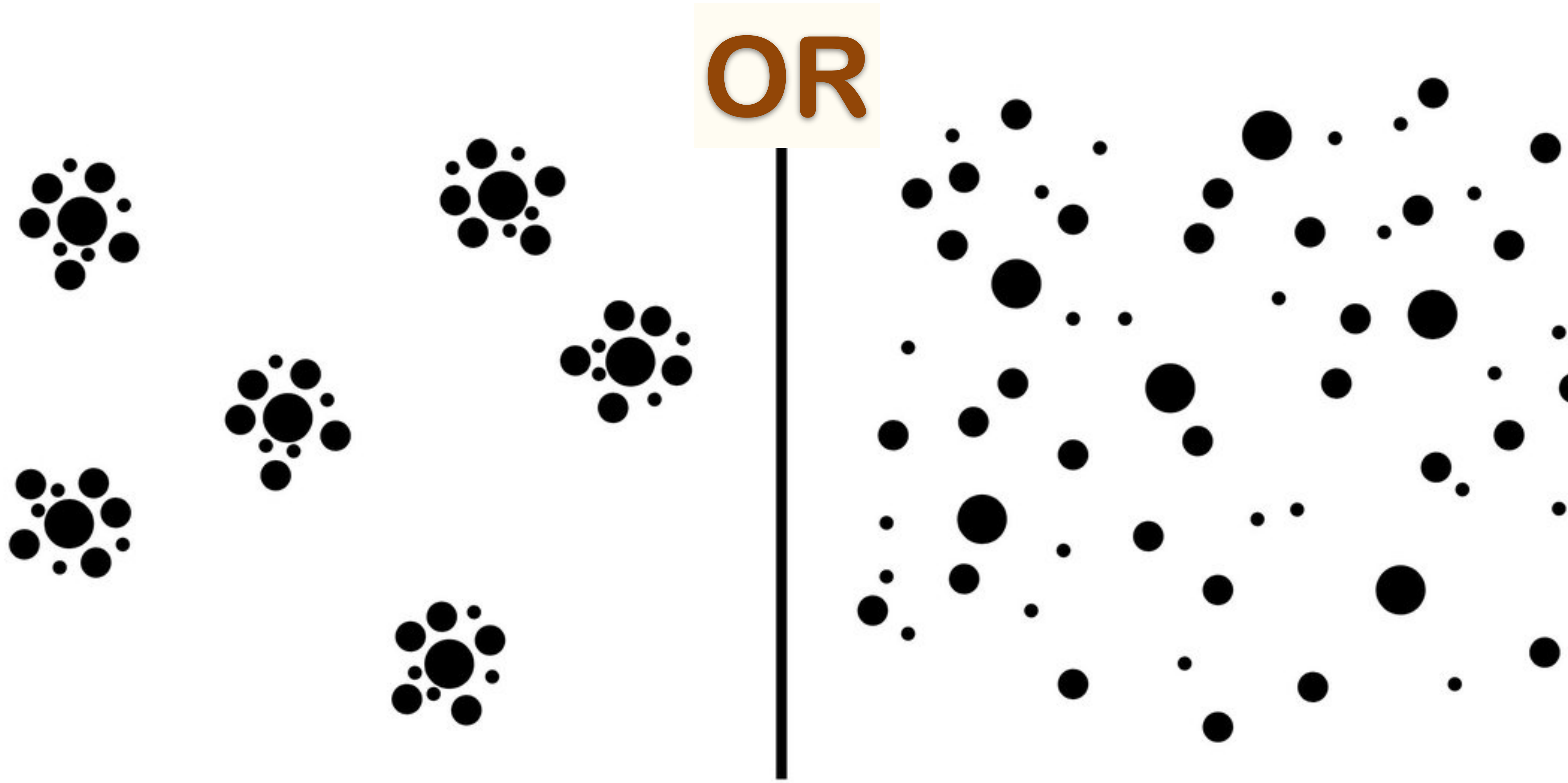


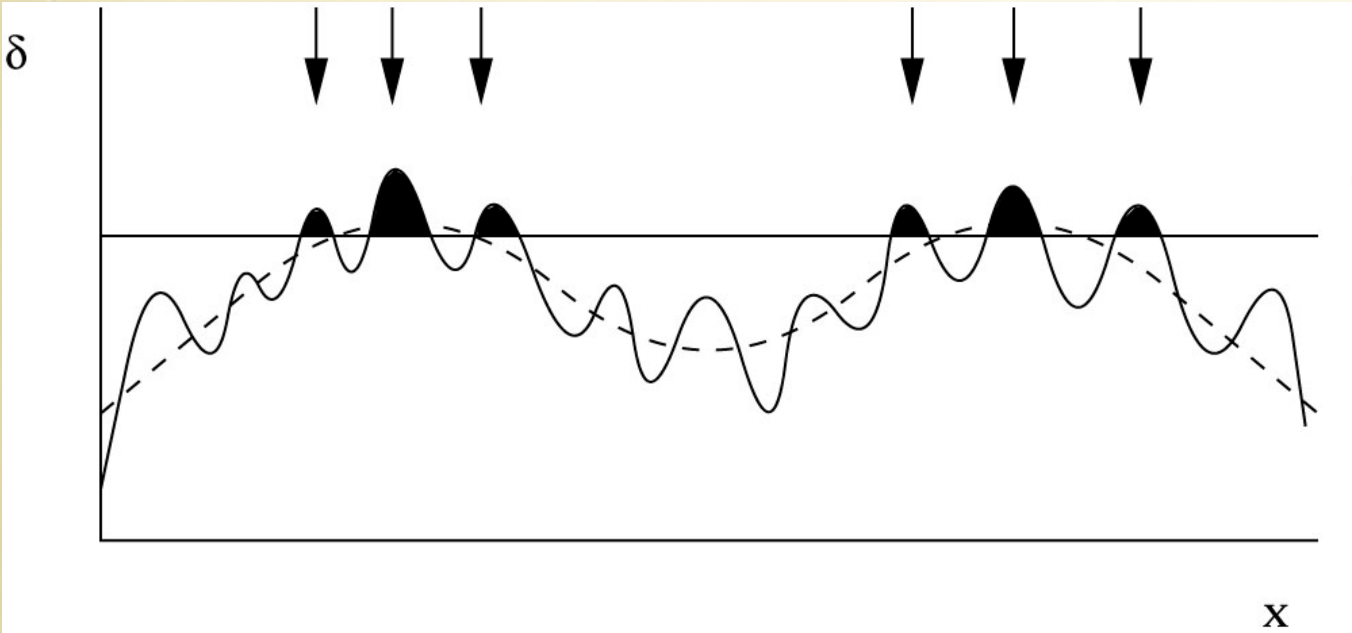
FIG. 1. Schematic representation of qualitatively different small-scale spatial distribution of PBHs at formation. On the left, PBHs are in dense clusters, as predicted in Ref. [22]. On the right, PBHs are distributed approximately randomly. In this work, we show that the latter distribution is what is expected for PBHs forming from large density fluctuations.

Ali-Haimoud, PRL 121, 081304 (2018)

Simple model of peak clustering

A simple model of peak clustering

- Thresholded regions are proxy for peaks



$$\delta_{\rm th} = \nu\sigma$$

$$\rho_{\rm pk}(\boldsymbol{x}) \sim \rho_{\rm th}(\boldsymbol{x}) = \bar{\rho}_{\rm tot}\Theta(\delta(\boldsymbol{x}) - \delta_{\rm th})$$

$$\left[\nu \equiv \frac{\delta_{\rm th}}{\sigma}, \quad \sigma^2 \equiv \langle\delta^2\rangle\right]$$

$$P(\delta) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\delta^2/\sigma^2} \quad \rightarrow \quad \beta \equiv \frac{\bar{\rho}_{\rm th}}{\bar{\rho}_{\rm tot}} = \int d\delta P(\delta)\Theta(\delta - \delta_{\rm th}) = \frac{1}{2}{\rm erfc}\left(\frac{\nu}{\sqrt{2}}\right)$$

$$1 + \xi_{\rm th}(r) = \frac{\langle\rho_{\rm th}(\boldsymbol{x}_1)\rho_{\rm th}(\boldsymbol{x}_2)\rangle}{\bar{\rho}_{\rm th}^2} = \frac{1}{\beta^2}\int_\nu^\infty \frac{d\nu_1 d\nu_2}{2\pi}\frac{1}{\sqrt{1 - w^2(r)}}\exp\left[-\frac{\nu_1^2 + \nu_2^2 - 2w(r)\nu_1\nu_2}{2(1 - w^2(r))}\right]$$

Kaiser, ApJL 284, L9 (1984)

$$\left[r = |\boldsymbol{x}_1 - \boldsymbol{x}_2|, \quad w(r) = \frac{\xi(r)}{\sigma^2}, \quad \xi(r) = \langle\delta(\boldsymbol{x}_1)\delta(\boldsymbol{x}_2)\rangle\right]$$

Limiting cases

- large-separation limit

Politzer and Wise, ApJL 285, L1 (1984)

$$1+\xi_{\mathrm{pk}}(r)\approx e^{w(r)\nu^2},\quad w(r)\ll\nu^{-1}\ll 1$$

- &

Kaiser, ApJL 284, L9 (1984)

$$\xi_{\mathrm{pk}}(r)\approx\nu^2 w(r)=\left(\frac{\nu}{\sigma}\right)^2\xi(r),\quad w(r)\ll\nu^{-2}\ll 1$$

  - (enhancement of correlation function)

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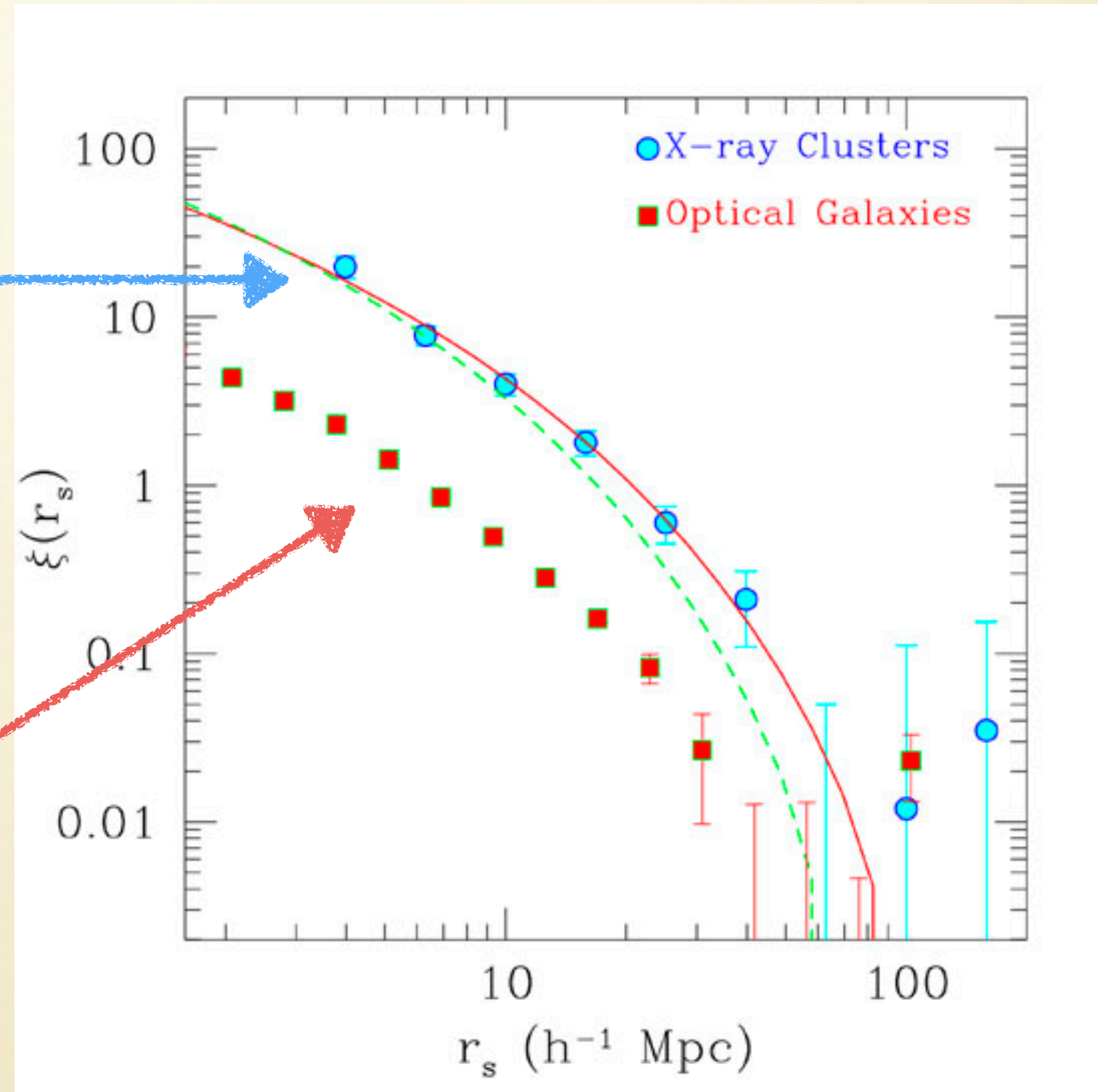
Enhancement of correlation function



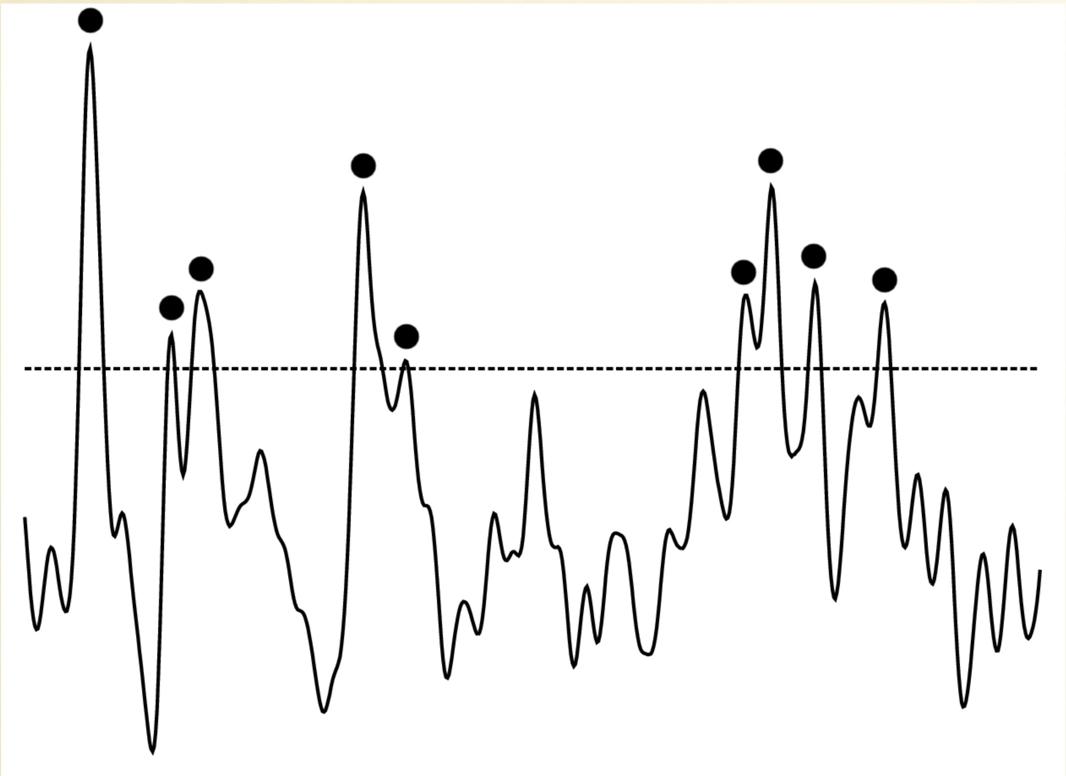
Borgani and Guzzo, Nature 409, 39 (2001)

Higher-order theory of peak clustering

Peak theory

BBKS:
Bardeen, Bond, Kaiser and Szalay, ApJ 304, 15 (1986)

- Thresholded regions ≠ Peaks



$$n_{\rm pk} = \Theta(\alpha - \nu)\delta_{\rm D}^3(\alpha_{,i})\Theta(\lambda_3)|\det(\alpha_{,ij})|$$

$$\alpha = \delta/\sigma, \quad \alpha_{,i} = \partial_i \alpha, \quad \alpha_{,ij} = \partial_i \partial_j \alpha$$

$$(\lambda_1 \geq \lambda_2 \geq \lambda_3 \ : \ \text{eigenvalues of } \ -\alpha_{,ij})$$

- $$\bar{n}_{\rm pk}(\nu) = \frac{1}{2(2\pi)^2 {R_*}^3}\int_0^\infty dx\, f(x)\, e^{-x^2/2}\, \mathrm{erfc}\left[\frac{\nu - \gamma x}{\sqrt{2(1-\gamma^2)}}\right]$$

$$f(x) = \frac{x}{2}(x^2 - 3)\left[\mathrm{erf}\left(\frac{1}{2}\sqrt{\frac{5}{2}}\,x\right) + \mathrm{erf}\left(\sqrt{\frac{5}{2}}\,x\right)\right] + \sqrt{\frac{2}{5\pi}}\left[\left(\frac{x^2}{2} - \frac{8}{5}\right)e^{-5x^2/2} + \left(\frac{31}{4}x^2 + \frac{8}{5}\right)e^{-5x^2/8}\right]$$

Peak clustering

- Exact evaluation of the peak clustering is difficult
  - BBKS derived approximate peak clustering in a large-separation limit (w/o exclusion effect etc.)
- Correlation function of peaks:

$$\langle n_{\rm pk}(\boldsymbol{x}_1) n_{\rm pk}(\boldsymbol{x}_2)\rangle =$$

$$\langle \Theta(\alpha(\boldsymbol{x}_1)-\nu)\delta^3_{\rm D}(\alpha_{,i}(\boldsymbol{x}_1))\Theta(\lambda_3(\boldsymbol{x}_1)|\det(\alpha_{,ij}(\boldsymbol{x}_1))|\Theta(\alpha(\boldsymbol{x}_2)-\nu)\delta^3_{\rm D}(\alpha_{,i}(\boldsymbol{x}_2))\Theta(\lambda_3(\boldsymbol{x}_2)|\det(\alpha_{,ij}(\boldsymbol{x}_2))|\rangle$$

  - $\Longrightarrow$ difficult !!
    - Naively 20-dimensional integrations !!

Peak clustering

- iPT formalism of bias

Matsubara ApJS 101, 1 (1995)
Matsubara PRD 83, 083518 (2011)
Lazeyras et al. PRD 93, 063007 (2016)

$$P_{\rm pk}(k) = \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\boldsymbol{k}_{1\cdots n}=\boldsymbol{k}} [\hat{c}_n(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_n)]^2 W^2(k_1 R) \cdots W^2(k_n R) P(k_1) \cdots P(k_n)$$

$$\hat{c}_1(\boldsymbol{k}) = b_{10} + b_{01} k^2$$

$$\hat{c}_2(\boldsymbol{k}_1, \boldsymbol{k}_2) = b_{20} + b_{11}\left({k_1}^2 + {k_2}^2\right) + b_{02}{k_1}^2{k_2}^2 - 2\chi_1(\boldsymbol{k}_1 \cdot \boldsymbol{k}_2) + \omega_{10}\left[3(\boldsymbol{k}_1 \cdot \boldsymbol{k}_2)^2 - {k_1}^2{k_2}^2\right]$$

$$\begin{aligned}\hat{c}_3(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) = {} & b_{30} + b_{21}\left({k_1}^2 + {k_2}^2 + {k_3}^2\right) + b_{12}\left({k_1}^2{k_2}^2 + \text{cyc.}\right) + b_{03}{k_1}^2{k_2}^2{k_3}^2 - 2b_{10}\chi_1(\boldsymbol{k}_1 \cdot \boldsymbol{k}_2 + \text{cyc.}) \\ & - 2b_{01}\chi_1\left[(\boldsymbol{k}_1 \cdot \boldsymbol{k}_2){k_3}^2 + \text{cyc.}\right] + c_{10010}\left\{\left[3(\boldsymbol{k}_1 \cdot \boldsymbol{k}_2)^2 - {k_1}^2{k_2}^2\right] + \text{cyc.}\right\} \\ & + 3c_{01010}\left[(\boldsymbol{k}_1 \cdot \boldsymbol{k}_2)^2{k_3}^2 + \text{cyc.} - {k_1}^2{k_2}^2{k_3}^2\right] \\ & - 3\varpi_{01}\left[(\boldsymbol{k}_1 \cdot \boldsymbol{k}_2)(\boldsymbol{k}_2 \cdot \boldsymbol{k}_3)(\boldsymbol{k}_3 \cdot \boldsymbol{k}_1) - \frac{1}{3}\left[(\boldsymbol{k}_1 \cdot \boldsymbol{k}_2)^2{k_3}^2 + \text{cyc.}\right] + \frac{2}{9}{k_1}^2{k_2}^2{k_3}^2\right]\end{aligned}$$

$$\int_{\boldsymbol{k}_{1\cdots n}=\boldsymbol{k}} \cdots \equiv \int \frac{d^3k_1}{(2\pi)^3} \cdots \frac{d^3k_n}{(2\pi)^3} (2\pi)^3 \delta_{\rm D}^3(\boldsymbol{k}_{1\cdots n} - \boldsymbol{k}) \cdots$$

$$\boldsymbol{k}_{1\cdots n} \equiv \boldsymbol{k}_1 + \cdots + \boldsymbol{k}_n$$

Peak clustering

- Reducing multi-dimensional integrals
  - Rayleigh expansion & analytic angular integrations → combination of 1D FFT
    - Schmittfull & Blah 2016, Schmittfull+ 2016
    - Using Mathematica is critical
    - 2nd order result agrees with a direct calculation by Desjacques et al. (2010)

$$\xi_{\rm pk}(r) = \int \frac{k^2 dk}{2\pi^2} j_0(kr) P_{\rm pk}(k)$$

$$\xi_{\rm pk}(r) = \sum_{N=1}^{\infty} \frac{1}{N!} \xi_{\rm pk}^{(N)}(r),$$

$$\xi_l^{(n)}(r) \equiv \int \frac{k^2 dk}{2\pi^2} k^n j_l(kr) P_{\rm L}(k) W^2(kR)$$

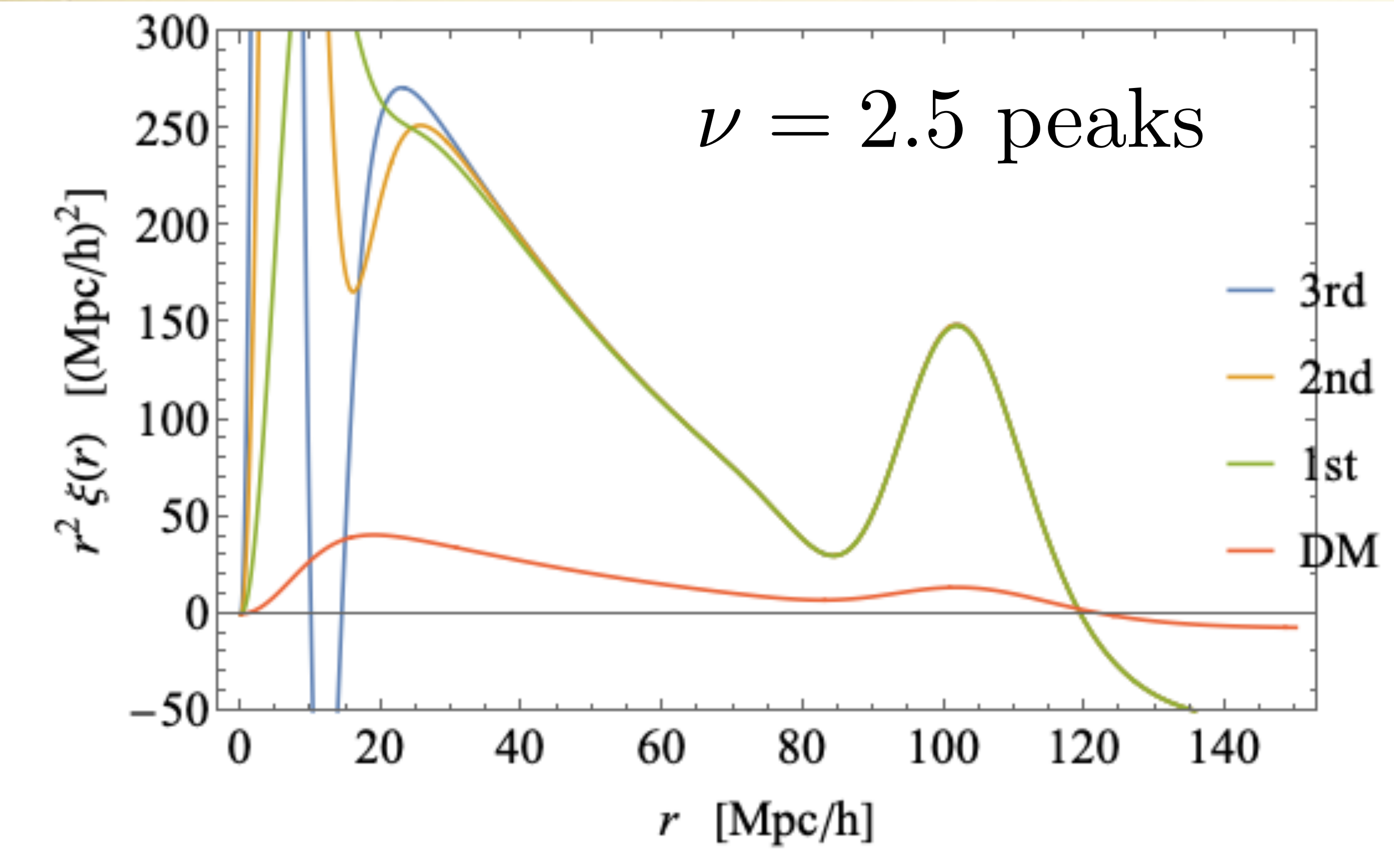
$$\xi_{\rm pk}^{(1)}(r) = b_{10}{}^2 \xi_0^{(0)}(r) + 2 b_{10} b_{01} \xi_0^{(2)}(r) + b_{01}{}^2 \xi_0^{(4)}(r), \tag{A4}$$

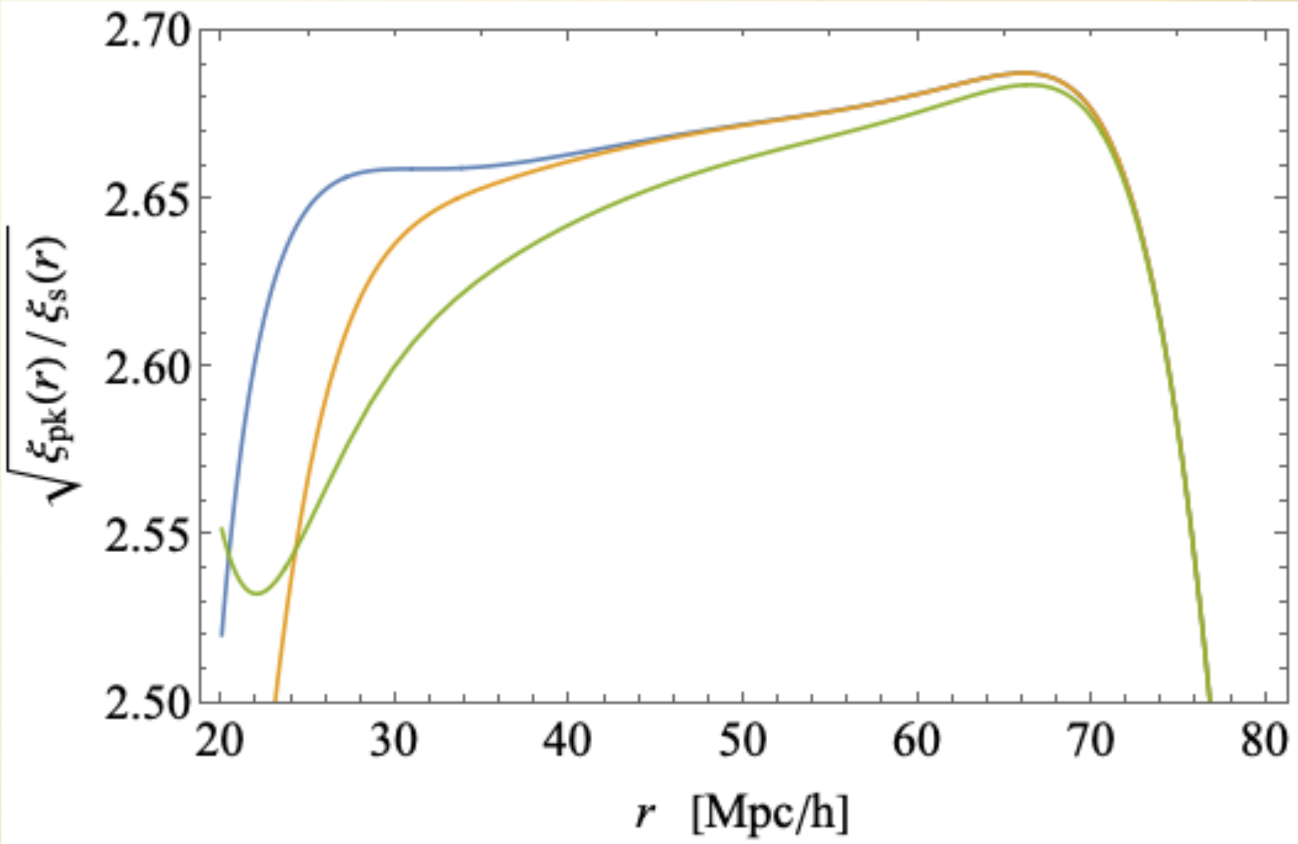
$$\begin{aligned}
\xi_{\rm pk}^{(2)} = {} & b_{20}{}^2 \left(\xi_0^{(0)}\right)^2 + 4 b_{20} b_{11} \xi_0^{(0)} \xi_0^{(2)} + 2 b_{11}{}^2 \xi_0^{(0)} \xi_0^{(4)} + 2\left(b_{20} b_{02} + b_{11}{}^2 + \frac{2}{3}\chi_1{}^2\right)\left(\xi_0^{(2)}\right)^2 + 4 b_{11} b_{02} \xi_0^{(2)} \xi_0^{(4)} + 4 b_{20} \chi_1 \left(\xi_1^{(1)}\right)^2 \\
& + 8 b_{11} \chi_1 \xi_1^{(1)} \xi_1^{(3)} + \left(b_{02}{}^2 + \frac{4}{5}\omega_{10}{}^2\right)\left(\xi_0^{(4)}\right)^2 + 4\left(b_{02} + \frac{4}{5}\omega_{10}\right)\chi_1 \left(\xi_1^{(3)}\right)^2 + 4\left(b_{20}\omega_{10} + \frac{2}{3}\chi_1{}^2\right)\left(\xi_2^{(2)}\right)^2 + 8 b_{11}\omega_{10} \xi_2^{(2)} \xi_2^{(4)} \\
& + 4\left(b_{02} + \frac{2}{7}\omega_{10}\right)\omega_{10}\left(\xi_2^{(4)}\right)^2 + \frac{24}{5}\chi_1\omega_{10}\left(\xi_3^{(3)}\right)^2 + \frac{72}{35}\omega_{10}{}^2\left(\xi_4^{(4)}\right)^2,
\end{aligned} \tag{A5}$$

$$\begin{aligned}
\xi_{\rm pk}^{(3)} = {} & b_{30}{}^2 \left(\xi_0^{(0)}\right)^3 + 6 b_{21} b_{30} \xi_0^{(2)} \left(\xi_0^{(0)}\right)^2 + 3 b_{21}{}^2 \xi_0^{(4)} \left(\xi_0^{(0)}\right)^2 + 12 b_{10} b_{30} \chi_1 \left(\xi_1^{(1)}\right)^2 \xi_0^{(0)} + 2\left(3 b_{21}{}^2 + 2 b_{10}{}^2 \chi_1{}^2 + 3 b_{12} b_{30}\right)\left(\xi_0^{(2)}\right)^2 \xi_0^{(0)} \\
& + 4\left(2 b_{10}{}^2 \chi_1{}^2 + 3 b_{30} c_{10010}\right)\left(\xi_2^{(2)}\right)^2 \xi_0^{(0)} + 12\chi_1\left(b_{10} b_{12} + \frac{4}{5} b_{10} c_{10010}\right)\left(\xi_1^{(3)}\right)^2 \xi_0^{(0)} + \frac{72}{5} b_{10}\chi_1 c_{10010}\left(\xi_3^{(3)}\right)^2 \xi_0^{(0)} \\
& + 3\left(b_{12}{}^2 + \frac{4}{5} c_{10010}{}^2\right)\left(\xi_0^{(4)}\right)^2 \xi_0^{(0)} + 12\left(\frac{2}{7} c_{10010} + b_{12}\right) c_{10010}\left(\xi_2^{(4)}\right)^2 \xi_0^{(0)} + \frac{216}{35} c_{10010}{}^2 \left(\xi_4^{(4)}\right)^2 \xi_0^{(0)} + 24 b_{10} b_{21} \chi_1 \xi_1^{(1)} \xi_1^{(3)} \xi_0^{(0)} \\
& + 12 b_{12} b_{21} \xi_0^{(2)} \xi_0^{(4)} \xi_0^{(0)} + 24 b_{21} c_{10010} \xi_2^{(2)} \xi_2^{(4)} \xi_0^{(0)} + 2\left(4 b_{01} b_{10} \chi_1{}^2 + 3 b_{12} b_{21} + b_{03} b_{30}\right)\left(\xi_0^{(2)}\right)^3 + \frac{4}{3} b_{30}\varpi_{01}\left(\xi_2^{(2)}\right)^3 \\
& + \left(b_{03}{}^2 + \frac{12}{5} c_{01010}{}^2 + \frac{14}{225}\varpi_{01}{}^2\right)\left(\xi_0^{(4)}\right)^3 - 4\left(\frac{12}{7} c_{01010}{}^2 + \frac{41}{3087}\varpi_{01}{}^2 - \frac{1}{3} b_{03}\varpi_{01} + \frac{6}{49} c_{01010}\varpi_{01}\right)\left(\xi_2^{(4)}\right)^3 \\
& + \frac{1296}{8575}\varpi_{01}{}^2\left(\xi_4^{(4)}\right)^3 + 4\left(4 b_{01} b_{10}\chi_1{}^2 + 3 b_{30} c_{01010} + 3 b_{21} c_{10010}\right)\xi_0^{(2)}\left(\xi_2^{(2)}\right)^2 + 4\chi_1\left(3 b_{03} b_{10} + 3 b_{01} b_{12} - 2 b_{01}{}^2\chi_1\right)\xi_0^{(2)}\left(\xi_1^{(3)}\right)^2 \\
& + \frac{48}{5}\chi_1\left(b_{10} c_{01010} + b_{01} c_{10010}\right)\xi_0^{(2)}\left(\xi_1^{(3)}\right)^2 + 8\chi_1\left(2 b_{01}{}^2\chi_1 - \frac{12}{5} b_{10} c_{01010} + \frac{7}{25} b_{10}\varpi_{01}\right)\xi_2^{(2)}\left(\xi_1^{(3)}\right)^2 \\
& + \frac{72}{5}\chi_1\left(b_{10} c_{01010} + b_{01} c_{10010}\right)\xi_0^{(2)}\left(\xi_3^{(3)}\right)^2 + \frac{96}{25} b_{10}\chi_1\varpi_{01}\xi_2^{(2)}\left(\xi_3^{(3)}\right)^2 + 6\left(b_{03} b_{12} + \frac{4}{5} c_{01010} c_{10010}\right)\xi_0^{(2)}\left(\xi_0^{(4)}\right)^2 \\
& + 12\left(b_{12} c_{01010} + b_{03} c_{10010} + \frac{4}{7} c_{01010} c_{10010}\right)\xi_0^{(2)}\left(\xi_2^{(4)}\right)^2 + 4\left(b_{12}\varpi_{01} - \frac{24}{7} c_{01010} c_{10010} - \frac{6}{49} c_{10010}\varpi_{01}\right)\xi_2^{(2)}\left(\xi_2^{(4)}\right)^2 \\
& + 2\left(6 b_{03} c_{01010} + \frac{144}{35} c_{01010}{}^2 - \frac{1}{35}\varpi_{01}{}^2 - \frac{8}{5} c_{01010}\varpi_{01}\right)\xi_0^{(4)}\left(\xi_2^{(4)}\right)^2 + \frac{432}{35} c_{01010} c_{10010}\xi_0^{(2)}\left(\xi_4^{(4)}\right)^2 + \frac{144}{49} c_{10010}\varpi_{01}\xi_2^{(2)}\left(\xi_4^{(4)}\right)^2 \\
& + \frac{8}{35}\left(27 c_{01010}{}^2 + \frac{3}{5}\varpi_{01}{}^2\right)\xi_0^{(4)}\left(\xi_4^{(4)}\right)^2 + \frac{48}{49}\varpi_{01}\left(\frac{1}{7}\varpi_{01} + 3 c_{01010}\right)\xi_2^{(4)}\left(\xi_4^{(4)}\right)^2 \\
& + 4\chi_1\left(3 b_{10} b_{21} + 4 b_{01} b_{30} - 2 b_{10}{}^2\chi_1\right)\left(\xi_1^{(1)}\right)^2 \xi_0^{(2)} + 16 b_{10}{}^2\chi_1{}^2\left(\xi_1^{(1)}\right)^2 \xi_2^{(2)} + 8\chi_1\left(3 b_{10} b_{12} + 3 b_{01} b_{21} - 2 b_{01} b_{10}\chi_1\right)\xi_1^{(1)}\xi_0^{(2)}\xi_1^{(3)} \\
& + 32\chi_1\left(b_{01} b_{10}\chi_1 - \frac{3}{5} b_{10} c_{10010}\right)\xi_1^{(1)}\xi_2^{(2)}\xi_1^{(3)} + \frac{144}{5} b_{10}\chi_1 c_{10010}\xi_1^{(1)}\xi_2^{(2)}\xi_3^{(3)} + \frac{144}{5} b_{10}\chi_1 c_{01010}\xi_2^{(2)}\xi_1^{(3)}\xi_3^{(3)} \\
& - \frac{48}{25} b_{10}\chi_1\varpi_{01}\xi_2^{(2)}\xi_1^{(3)}\xi_3^{(3)} + 12 b_{01}\chi_1 b_{21}\left(\xi_1^{(1)}\right)^2 \xi_0^{(4)} + 2\left(3 b_{12}{}^2 + 2 b_{01}{}^2\chi_1{}^2 + 3 b_{03} b_{21}\right)\left(\xi_0^{(2)}\right)^2 \xi_0^{(4)} \\
& + 4\left(\frac{6}{5} c_{10010}{}^2 + 2 b_{01}{}^2\chi_1{}^2 + 3 b_{21} c_{01010}\right)\left(\xi_2^{(2)}\right)^2 \xi_0^{(4)} + 12 b_{01}\chi_1\left(b_{03} + \frac{4}{5} c_{01010}\right)\left(\xi_1^{(3)}\right)^2 \xi_0^{(4)} + \frac{72}{5} b_{01}\chi_1 c_{01010}\left(\xi_3^{(3)}\right)^2 \xi_0^{(4)} \\
& + 24 b_{01} b_{12}\chi_1\xi_1^{(1)}\xi_1^{(3)}\xi_0^{(4)} + 4\left(b_{21}\varpi_{01} - \frac{12}{7} c_{10010}{}^2\right)\left(\xi_2^{(2)}\right)^2 \xi_2^{(4)} + \frac{8}{5} b_{01}\chi_1\left(\frac{7}{5}\varpi_{01} - 12 c_{01010}\right)\left(\xi_1^{(3)}\right)^2 \xi_2^{(4)} \\
& + \frac{96}{25} b_{01}\chi_1\varpi_{01}\left(\xi_3^{(3)}\right)^2 \xi_2^{(4)} + 24\left(b_{21} c_{01010} + b_{12} c_{10010}\right)\xi_0^{(2)}\xi_2^{(2)}\xi_2^{(4)} - \frac{96}{5} b_{01}\chi_1 c_{10010}\xi_1^{(1)}\xi_1^{(3)}\xi_2^{(4)} + \frac{144}{5} b_{01}\chi_1 c_{10010}\xi_1^{(1)}\xi_3^{(3)}\xi_2^{(4)} \\
& + \frac{48}{5} b_{01}\chi_1\left(3 c_{01010} - \frac{1}{5}\varpi_{01}\right)\xi_1^{(3)}\xi_3^{(3)}\xi_2^{(4)} + 8\left(3 b_{12} c_{01010} + \frac{6}{5} c_{01010} c_{10010} - \frac{2}{5} c_{10010}\varpi_{01}\right)\xi_2^{(2)}\xi_0^{(4)}\xi_2^{(4)} + \frac{432}{35} c_{10010}{}^2\left(\xi_2^{(2)}\right)^2 \xi_4^{(4)} \\
& + \frac{24}{35}\left(18 c_{01010}{}^2 + \frac{17}{49}\varpi_{01}{}^2 - \frac{24}{7} c_{01010}\varpi_{01}\right)\left(\xi_2^{(4)}\right)^2 \xi_4^{(4)} + \frac{288}{35} c_{10010}\left(3 c_{01010} - \frac{2}{7}\varpi_{01}\right)\xi_2^{(2)}\xi_2^{(4)}\xi_4^{(4)},
\end{aligned} \tag{A6}$$

Peak clustering: Large-scale structure (Preliminary)

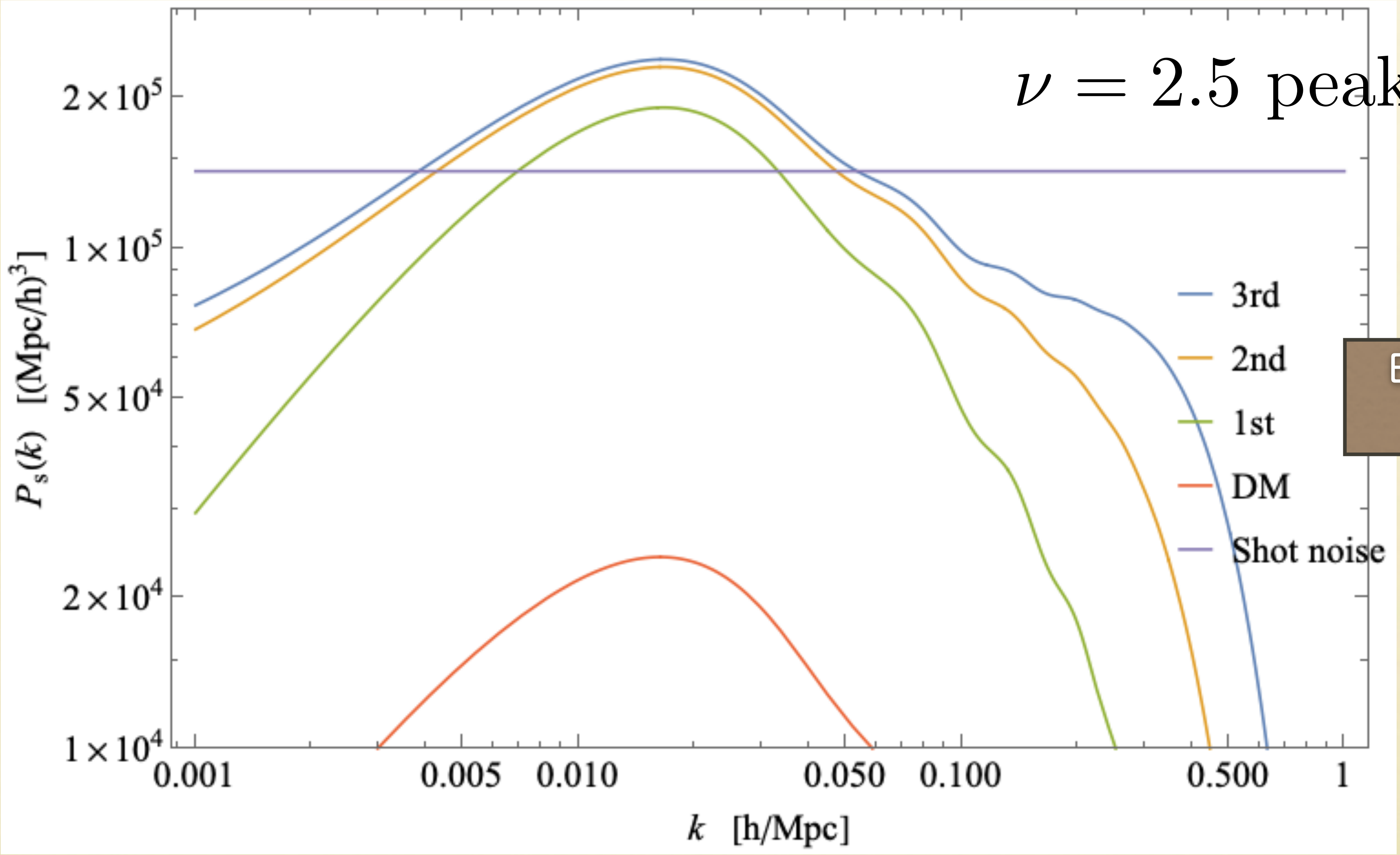
- Correlation function of peaks (CDM+baryons spectrum)





Peak clustering: Large-scale structure (Preliminary)

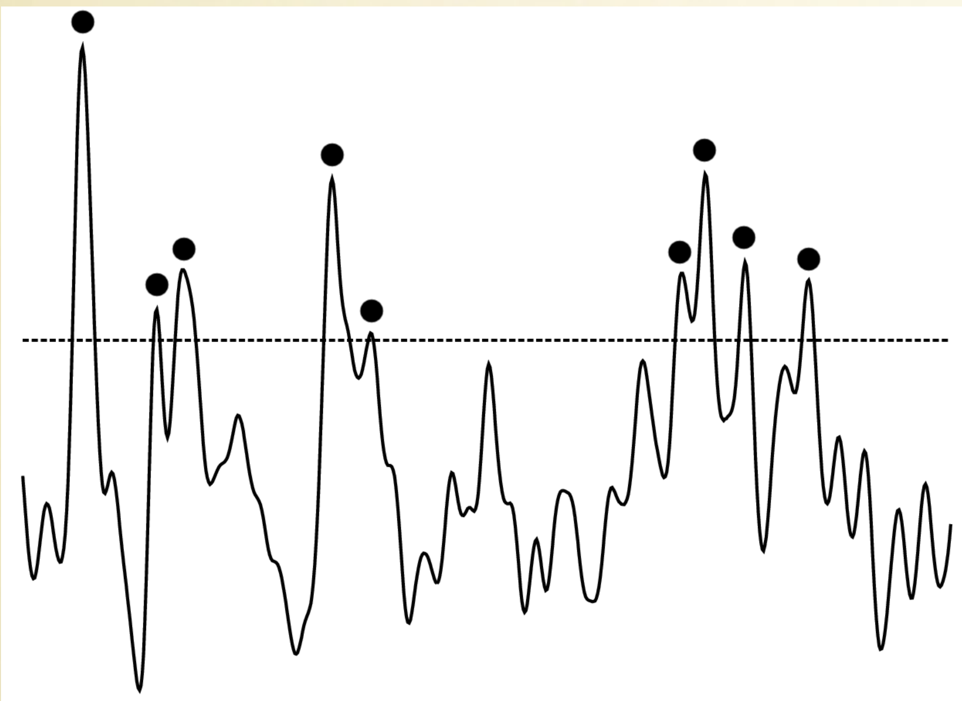
- Power spectrum of peaks (CDM+baryons spectrum)

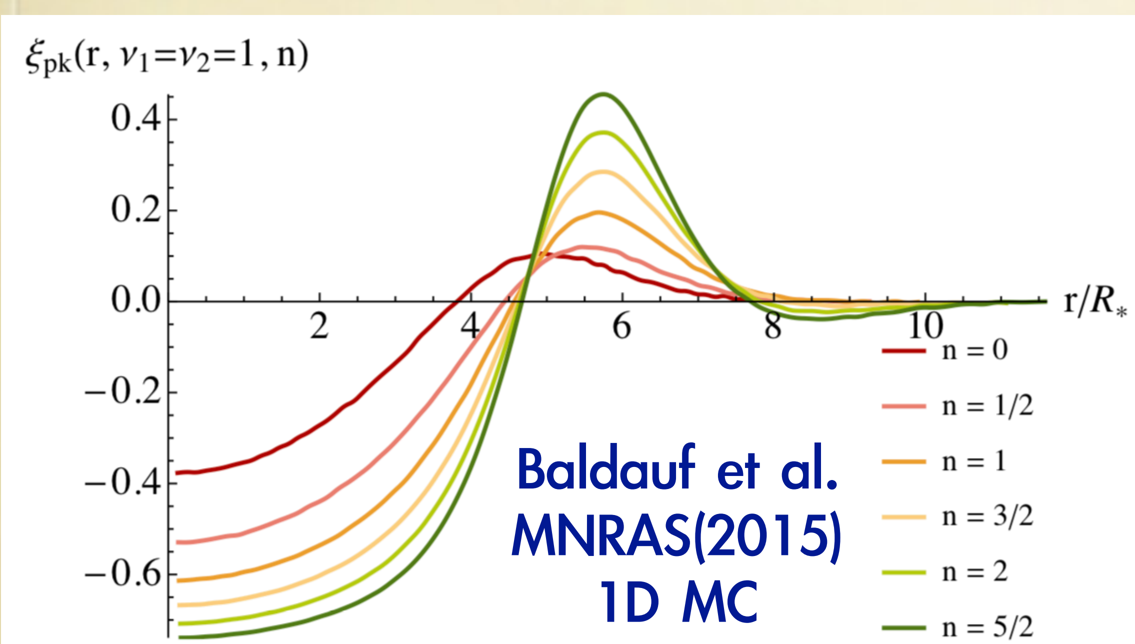


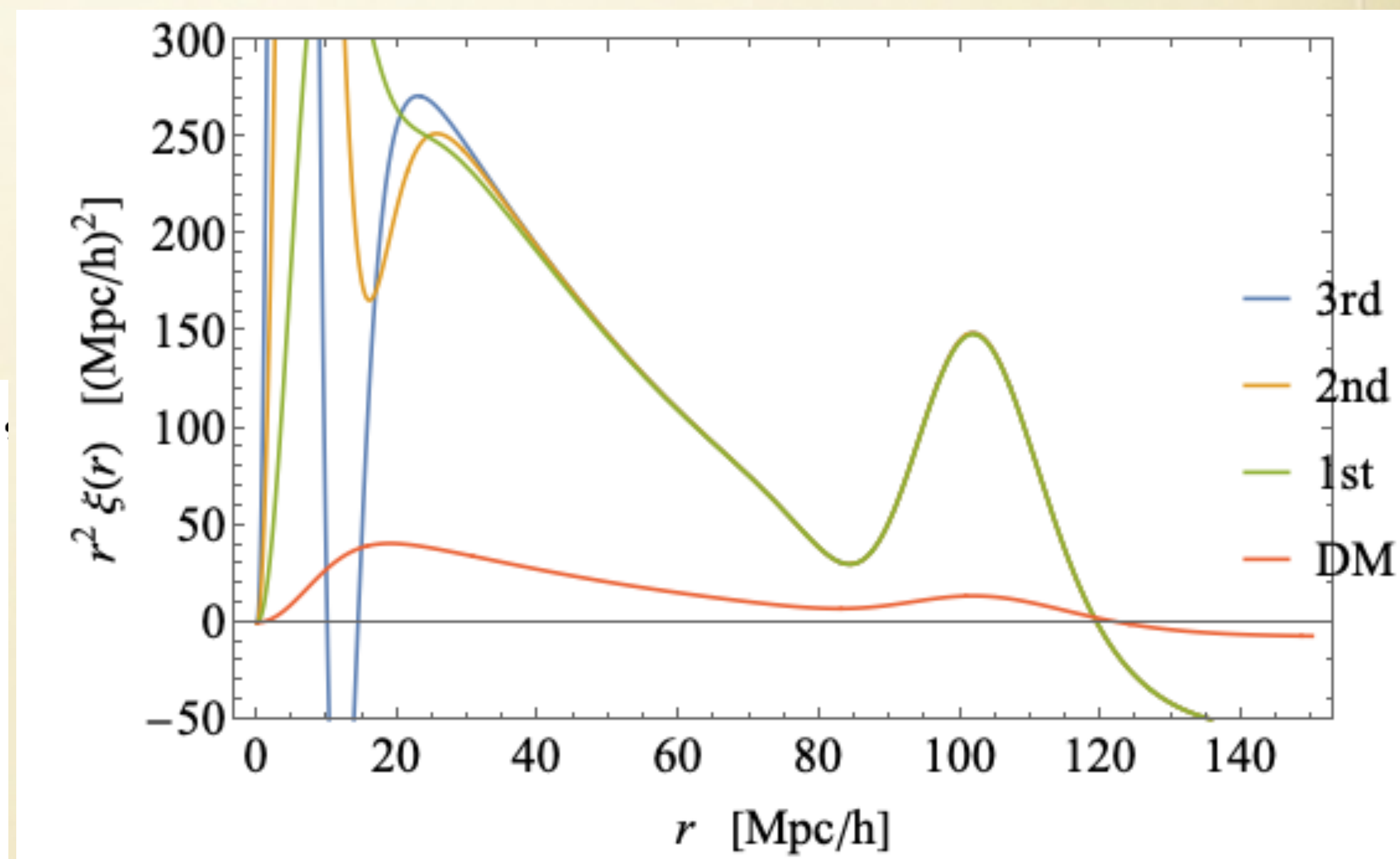
Expansion series are additive

Peak exclusion effect?

negative correlation
~ below 3 x smoothing scale







Summary

- Higher-order perturbation theory of peak clustering
- With iPT formalism, higher-order terms are systematically derived
- We need only 1D FFT to evaluate higher-order terms (while the number of terms are huge for higher orders, use of Mathematica helps)